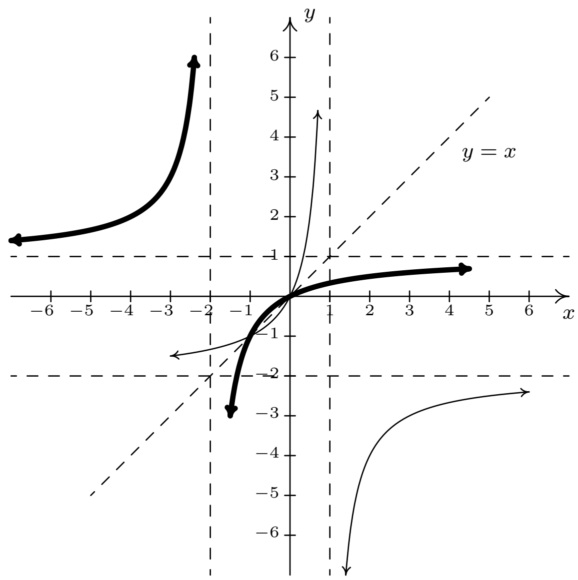
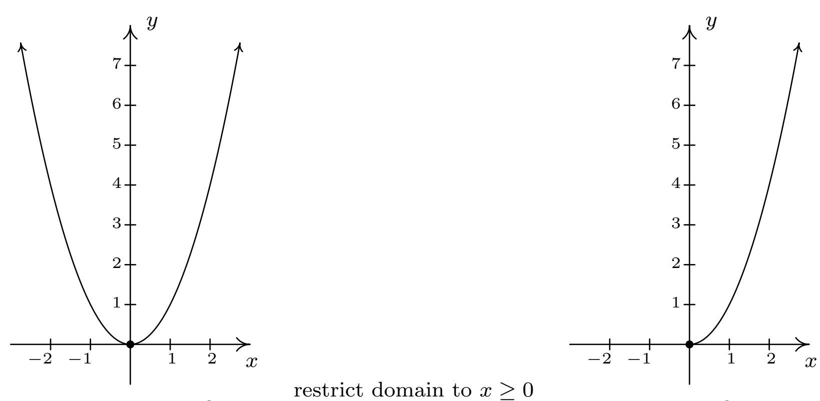


$y = g(x)$ and $\boldsymbol{y = g^{-1}(x)}$

□

We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible. However, if we restrict the domain of f, we can produce a new function g which is one-to-one. If we define $g(x) = x^2$, $x \geq 0$, then we have

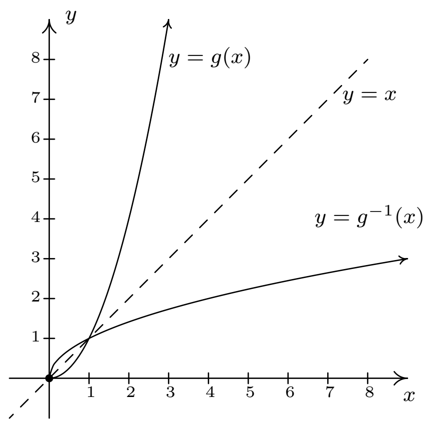


$y = f(x) = x^2$ $\xrightarrow{\text{restrict domain to } x \geq 0}$ $y = g(x) = x^2,\ x \geq 0$

The graph of g passes the Horizontal Line Test. To find an inverse of g, we proceed as usual

$$\begin{aligned} y &= g(x) \\ y &= x^2, \;\; x \geq 0 \\ x &= y^2, \;\; y \geq 0 \quad \text{switch } x \text{ and } y \\ y &= \pm\sqrt{x} \\ y &= \sqrt{x} \qquad\qquad \text{since } y \geq 0 \end{aligned}$$

We get $g^{-1}(x) = \sqrt{x}$. At first it looks like we'll run into the same trouble as before, but when we check the composition, the domain restriction on g saves the day. We get $\left(g^{-1} \circ g\right)(x) = g^{-1}(g(x)) = g^{-1}\left(x^2\right) = \sqrt{x^2} = |x| = x$, since $x \geq 0$. Checking $\left(g \circ g^{-1}\right)(x) = g\left(g^{-1}(x)\right) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$. Graphing[6] g and g^{-1} on the same set of axes shows that they are reflections about the line $y = x$.



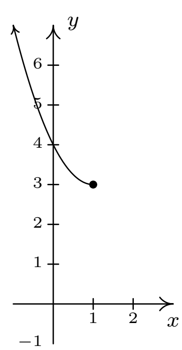
Our next example continues the theme of domain restriction.

Example 5.2.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(x) = \sqrt{x+2} - 1$

Solution.

1. The function j is a restriction of the function h from Example 5.2.1. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. We see that the graph of j passes the Horizontal Line Test and thus j is invertible.



$y = j(x)$

[6] We graphed $y = \sqrt{x}$ in Section 1.7.

We now use our algorithm[7] to find $j^{-1}(x)$.

$$\begin{array}{rcll}
y &=& j(x) & \\
y &=& x^2-2x+4, \;\; x\leq 1 & \\
x &=& y^2-2y+4, \;\; y\leq 1 & \text{switch } x \text{ and } y\\
0 &=& y^2-2y+4-x & \\
y &=& \dfrac{2\pm\sqrt{(-2)^2-4(1)(4-x)}}{2(1)} & \text{quadratic formula, } c=4-x\\
y &=& \dfrac{2\pm\sqrt{4x-12}}{2} & \\
y &=& \dfrac{2\pm\sqrt{4(x-3)}}{2} & \\
y &=& \dfrac{2\pm 2\sqrt{x-3}}{2} & \\
y &=& \dfrac{2\left(1\pm\sqrt{x-3}\right)}{2} & \\
y &=& 1\pm\sqrt{x-3} & \\
y &=& 1-\sqrt{x-3} & \text{since } y\leq 1.
\end{array}$$

We have $j^{-1}(x)=1-\sqrt{x-3}$. When we simplify $\left(j^{-1}\circ j\right)(x)$, we need to remember that the domain of j is $x\leq 1$.

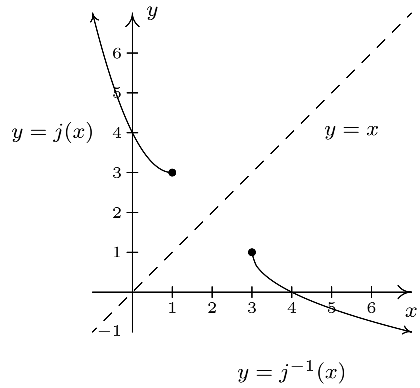
$$\begin{array}{rcll}
\left(j^{-1}\circ j\right)(x) &=& j^{-1}(j(x)) & \\
&=& j^{-1}\left(x^2-2x+4\right), \;\; x\leq 1 & \\
&=& 1-\sqrt{(x^2-2x+4)-3} & \\
&=& 1-\sqrt{x^2-2x+1} & \\
&=& 1-\sqrt{(x-1)^2} & \\
&=& 1-|x-1| & \\
&=& 1-(-(x-1)) & \text{since } x\leq 1\\
&=& x \checkmark &
\end{array}$$

Checking $j\circ j^{-1}$, we get

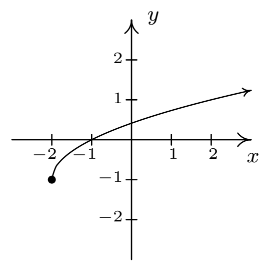
$$\begin{array}{rcl}
\left(j\circ j^{-1}\right)(x) &=& j\left(j^{-1}(x)\right)\\
&=& j\left(1-\sqrt{x-3}\right)\\
&=& \left(1-\sqrt{x-3}\right)^2-2\left(1-\sqrt{x-3}\right)+4\\
&=& 1-2\sqrt{x-3}+\left(\sqrt{x-3}\right)^2-2+2\sqrt{x-3}+4\\
&=& 3+x-3\\
&=& x \checkmark
\end{array}$$

[7] Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.

Using what we know from Section 1.7, we graph $y = j^{-1}(x)$ and $y = j(x)$ below.



2. We graph $y = k(x) = \sqrt{x+2} - 1$ using what we learned in Section 1.7 and see k is one-to-one.



$y = k(x)$

We now try to find k^{-1}.

$$\begin{array}{rcll} y & = & k(x) & \\ y & = & \sqrt{x+2} - 1 & \\ x & = & \sqrt{y+2} - 1 & \text{switch } x \text{ and } y \\ x+1 & = & \sqrt{y+2} & \\ (x+1)^2 & = & \left(\sqrt{y+2}\right)^2 & \\ x^2+2x+1 & = & y+2 & \\ y & = & x^2+2x-1 & \end{array}$$

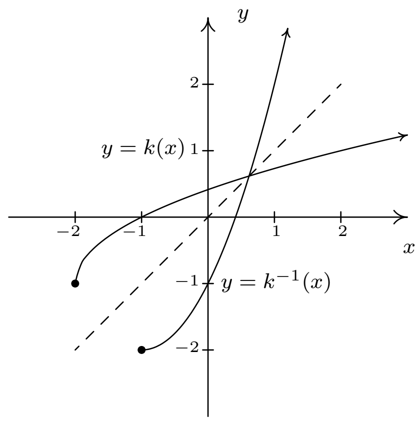
We have $k^{-1}(x) = x^2 + 2x - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted. Theorem 5.2 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $x \geq -1$. We now check that this works in our compositions.

$$
\begin{aligned}
\left(k^{-1} \circ k\right)(x) &= k^{-1}(k(x)) \\
&= k^{-1}\left(\sqrt{x+2}-1\right), \quad x \geq -2 \\
&= \left(\sqrt{x+2}-1\right)^2 + 2\left(\sqrt{x+2}-1\right) - 1 \\
&= \left(\sqrt{x+2}\right)^2 - 2\sqrt{x+2} + 1 + 2\sqrt{x+2} - 2 - 1 \\
&= x + 2 - 2 \\
&= x \checkmark
\end{aligned}
$$

and

$$
\begin{aligned}
\left(k \circ k^{-1}\right)(x) &= k\left(x^2 + 2x - 1\right) \quad x \geq -1 \\
&= \sqrt{(x^2 + 2x - 1) + 2} - 1 \\
&= \sqrt{x^2 + 2x + 1} - 1 \\
&= \sqrt{(x+1)^2} - 1 \\
&= |x+1| - 1 \\
&= x + 1 - 1 && \text{since } x \geq -1 \\
&= x \checkmark
\end{aligned}
$$

Graphically, everything checks out as well, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions.

Example 5.2.4. Recall from Section 2.1 that the price-demand equation for the PortaBoy game system is $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$, where x represents the number of systems sold weekly and p is the price per system in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.

2. Find and interpret $p^{-1}(220)$.

3. Recall from Section 2.3 that the weekly profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.

4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 2.3.3.

Solution.

1. We leave to the reader to show the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the weekly sales and furnished the price per system as the output, p^{-1} takes the price per system and returns the weekly sales as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in a week if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[8] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$. To understand what this means, recall that the original profit function P gave us the weekly profit as a function of the weekly sales. The function p^{-1} gives us the weekly sales as a function of the price. Hence, $P \circ p^{-1}$ takes as its input a price. The function p^{-1} returns the weekly sales, which in turn is fed into P to return the weekly profit. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the weekly profit (in dollars) as a function of the price per system, x, using the weekly sales $p^{-1}(x)$ as the 'middle man'.

4. We know from Section 2.3 that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation 2.4, $x = -\frac{b}{2a} = 165$. Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example 2.3.3, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

[8] It is good review to actually do this!

5.2.1 Exercises

In Exercises 1 - 20, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

1. $f(x) = 6x - 2$
2. $f(x) = 42 - x$
3. $f(x) = \dfrac{x-2}{3} + 4$
4. $f(x) = 1 - \dfrac{4+3x}{5}$
5. $f(x) = \sqrt{3x-1} + 5$
6. $f(x) = 2 - \sqrt{x-5}$
7. $f(x) = 3\sqrt{x-1} - 4$
8. $f(x) = 1 - 2\sqrt{2x+5}$
9. $f(x) = \sqrt[5]{3x-1}$
10. $f(x) = 3 - \sqrt[3]{x-2}$
11. $f(x) = x^2 - 10x,\ x \geq 5$
12. $f(x) = 3(x+4)^2 - 5,\ x \leq -4$
13. $f(x) = x^2 - 6x + 5,\ x \leq 3$
14. $f(x) = 4x^2 + 4x + 1,\ x < -1$
15. $f(x) = \dfrac{3}{4-x}$
16. $f(x) = \dfrac{x}{1-3x}$
17. $f(x) = \dfrac{2x-1}{3x+4}$
18. $f(x) = \dfrac{4x+2}{3x-6}$
19. $f(x) = \dfrac{-3x-2}{x+3}$
20. $f(x) = \dfrac{x-2}{2x-1}$

With help from your classmates, find the inverses of the functions in Exercises 21 - 24.

21. $f(x) = ax + b,\ a \neq 0$
22. $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$
23. $f(x) = ax^2 + bx + c$ where $a \neq 0$, $x \geq -\dfrac{b}{2a}$.
24. $f(x) = \dfrac{ax+b}{cx+d}$, (See Exercise 33 below.)
25. In Example 1.5.3, the price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

    (a) Find $p^{-1}(x)$ and state its domain.

    (b) Find and interpret $p^{-1}(105)$.

    (c) In Example 1.5.3, we determined that the profit (in dollars) made from producing and selling x dOpis per week is $P(x) = -15x^2 + 350x - 2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

26. Show that the Fahrenheit to Celsius conversion function found in Exercise 35 in Section 2.1 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

27. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Since finding a formula for its inverse is beyond the scope of this textbook, use Theorem 5.2 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$.

28. Let $f(x) = \frac{2x}{x^2-1}$. Using the techniques in Section 4.2, graph $y = f(x)$. Verify that f is one-to-one on the interval $(-1, 1)$. Use the procedure outlined on Page 384 and your graphing calculator to find the formula for $f^{-1}(x)$. Note that since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

29. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

30. If f is odd and invertible, prove that f^{-1} is also odd.

31. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

32. What graphical feature must a function f possess for it to be its own inverse?

33. What conditions must you place on the values of a, b, c and d in Exercise 24 in order to guarantee that the function is invertible?

5.2.2 Answers

1. $f^{-1}(x) = \dfrac{x+2}{6}$
2. $f^{-1}(x) = 42 - x$
3. $f^{-1}(x) = 3x - 10$
4. $f^{-1}(x) = -\frac{5}{3}x + \frac{1}{3}$
5. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}$, $x \geq 5$
6. $f^{-1}(x) = (x-2)^2 + 5,\ x \leq 2$
7. $f^{-1}(x) = \frac{1}{9}(x+4)^2 + 1$, $x \geq -4$
8. $f^{-1}(x) = \frac{1}{8}(x-1)^2 - \frac{5}{2}$, $x \leq 1$
9. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
10. $f^{-1}(x) = -(x-3)^3 + 2$
11. $f^{-1}(x) = 5 + \sqrt{x+25}$
12. $f^{-1}(x) = -\sqrt{\frac{x+5}{3}} - 4$
13. $f^{-1}(x) = 3 - \sqrt{x+4}$
14. $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$, $x > 1$
15. $f^{-1}(x) = \dfrac{4x-3}{x}$
16. $f^{-1}(x) = \dfrac{x}{3x+1}$
17. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
18. $f^{-1}(x) = \dfrac{6x+2}{3x-4}$
19. $f^{-1}(x) = \dfrac{-3x-2}{x+3}$
20. $f^{-1}(x) = \dfrac{x-2}{2x-1}$

25. (a) $p^{-1}(x) = \frac{450-x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$
    (b) $p^{-1}(105) = 23$. This means that if the price is set to \$105 then 23 dOpis will be sold.
    (c) $\left(P \circ p^{-1}\right)(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000$, $0 \leq x \leq 450$. The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi at either \$285 or \$270, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.

27. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

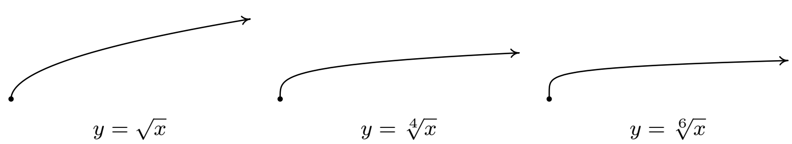
5.3 Other Algebraic Functions

This section serves as a watershed for functions which are combinations of polynomial, and more generally, rational functions, with the operations of radicals. It is business of Calculus to discuss these functions in all the detail they demand so our aim in this section is to help shore up the requisite skills needed so that the reader can answer Calculus's call when the time comes. We briefly recall the definition and some of the basic properties of radicals from Intermediate Algebra.[1]

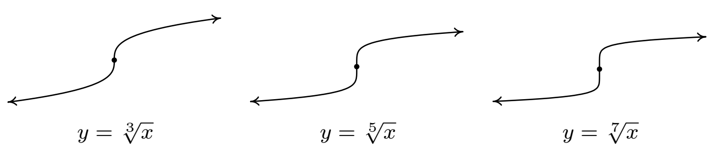
Definition 5.4. Let x be a real number and n a natural number.[a] If n is odd, the **principal n^{th} root** of x, denoted $\sqrt[n]{x}$ is the unique real number satisfying $\left(\sqrt[n]{x}\right)^n = x$. If n is even, $\sqrt[n]{x}$ is defined similarly[b] provided $x \geq 0$ and $\sqrt[n]{x} \geq 0$. The **index** is the number n and the **radicand** is the number x. For $n = 2$, we write $\sqrt{x}$ instead of $\sqrt[2]{x}$.

[a] Recall this means $n = 1, 2, 3, \ldots$.
[b] Recall both $x = -2$ and $x = 2$ satisfy $x^4 = 16$, but $\sqrt[4]{16} = 2$, not -2.

It is worth remarking that, in light of Section 5.2, we could define $f(x) = \sqrt[n]{x}$ functionally as the inverse of $g(x) = x^n$ with the stipulation that when n is even, the domain of g is restricted to $[0, \infty)$. From what we know about $g(x) = x^n$ from Section 3.1 along with Theorem 5.3, we can produce the graphs of $f(x) = \sqrt[n]{x}$ by reflecting the graphs of $g(x) = x^n$ across the line $y = x$. Below are the graphs of $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$. The point $(0, 0)$ is indicated as a reference. The axes are hidden so we can see the vertical steepening near $x = 0$ and the horizontal flattening as $x \to \infty$.



The odd-indexed radical functions also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$. In the exercises, you'll have a chance to graph some basic radical functions using the techniques presented in Section 1.7.



We have used all of the following properties at some point in the textbook for the case $n = 2$ (the square root), but we list them here in generality for completeness.

[1] Although we discussed imaginary numbers in Section 3.4, we restrict our attention to real numbers in this section. See the epilogue on page 294 for more details.

Theorem 5.6. Properties of Radicals: Let x and y be real numbers and m and n be natural numbers. If $\sqrt[n]{x}$, $\sqrt[n]{y}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$
- **Powers of Radicals:** $\sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$
- **Quotient Rule:** $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$, provided $y \neq 0$.
- If n is odd, $\sqrt[n]{x^n} = x$; if n is even, $\sqrt[n]{x^n} = |x|$.

The proof of Theorem 5.6 is based on the definition of the principal roots and properties of exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{xy}$ is the unique real number such that $\left(\sqrt[n]{xy}\right)^n = xy$. Given that $\left(\sqrt[n]{x}\,\sqrt[n]{y}\right)^n = \left(\sqrt[n]{x}\right)^n \left(\sqrt[n]{y}\right)^n = xy$, it must be the case that $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. If n is even, then $\sqrt[n]{xy}$ is the unique non-negative real number such that $\left(\sqrt[n]{xy}\right)^n = xy$. Also note that since n is even, $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are also non-negative and hence so is $\sqrt[n]{x}\,\sqrt[n]{y}$. Proceeding as above, we find that $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{x}$ is a real number to start with.[2] The last property is an application of the power rule when n is odd, and the occurrence of the absolute value when n is even is due to the requirement that $\sqrt[n]{x} \geq 0$ in Definition 5.4. For instance, $\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2 = |-2|$, not -2. It's this last property which makes compositions of roots and powers delicate. This is especially true when we use exponential notation for radicals. Recall the following definition.

Definition 5.5. Let x be a real number, m an integer[a] and n a natural number.

- $x^{\frac{1}{n}} = \sqrt[n]{x}$ and is defined whenever $\sqrt[n]{x}$ is defined.
- $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$, whenever $\left(\sqrt[n]{x}\right)^m$ is defined.

[a]Recall this means $m = 0, \pm 1, \pm 2, \ldots$

The rational exponents defined in Definition 5.5 behave very similarly to the usual integer exponents from Elementary Algebra with one critical exception. Consider the expression $\left(x^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(x^{2/3}\right)^{3/2} = x^{\frac{2}{3}\cdot\frac{3}{2}} = x^1 = x$. However, if we substitute $x = -1$ and apply Definition 5.5, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. We see in this case that $\left(x^{2/3}\right)^{3/2} \neq x$. If we take the time to rewrite $\left(x^{2/3}\right)^{3/2}$ with radicals, we see

$$\left(x^{2/3}\right)^{3/2} = \left(\left(\sqrt[3]{x}\right)^2\right)^{3/2} = \left(\sqrt{\left(\sqrt[3]{x}\right)^2}\right)^3 = \left(\left|\sqrt[3]{x}\right|\right)^3 = \left|\left(\sqrt[3]{x}\right)^3\right| = |x|$$

[2]Otherwise we'd run into the same paradox we did in Section 3.4.

In the play-by-play analysis, we see that when we canceled the 2's in multiplying $\frac{2}{3} \cdot \frac{3}{2}$, we were, in fact, attempting to cancel a square with a square root. The fact that $\sqrt{x^2} = |x|$ and not simply x is the root[3] of the trouble. It may amuse the reader to know that $\left(x^{3/2}\right)^{2/3} = x$, and this verification is left as an exercise. The moral of the story is that when simplifying fractional exponents, it's usually best to rewrite them as radicals.[4] The last major property we will state, and leave to Calculus to prove, is that radical functions are continuous on their domains, so the Intermediate Value Theorem, Theorem 3.1, applies. This means that if we take combinations of radical functions with polynomial and rational functions to form what the authors consider the **algebraic functions**,[5] we can make sign diagrams using the procedure set forth in Section 4.2.

> **Steps for Constructing a Sign Diagram for an Algebraic Function**
>
> Suppose f is an algebraic function.
>
> 1. Place any values excluded from the domain of f on the number line with an '‽' above them.
> 2. Find the zeros of f and place them on the number line with the number 0 above them.
> 3. Choose a test value in each of the intervals determined in steps 1 and 2.
> 4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.

Our next example reviews quite a bit of Intermediate Algebra and demonstrates some of the new features of these graphs.

Example 5.3.1. For the following functions, state their domains and create sign diagrams. Check your answer graphically using your calculator.

1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(x) = \sqrt{2 - \sqrt[4]{x+3}}$
3. $h(x) = \sqrt[3]{\dfrac{8x}{x+1}}$
4. $k(x) = \dfrac{2x}{\sqrt{x^2-1}}$

Solution.

1. As far as domain is concerned, $f(x)$ has no denominators and no even roots, which means its domain is $(-\infty, \infty)$. To create the sign diagram, we find the zeros of f.

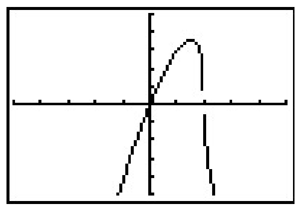
[3] Did you like that pun?
[4] In most other cases, though, rational exponents are preferred.
[5] As mentioned in Section 2.2, $f(x) = \sqrt{x^2} = |x|$ so that absolute value is also considered an algebraic function.

$$\begin{array}{rcl} f(x) & = & 0 \\ 3x\sqrt[3]{2-x} & = & 0 \\ 3x = 0 & \text{or} & \sqrt[3]{2-x} = 0 \\ x = 0 & \text{or} & \left(\sqrt[3]{2-x}\right)^3 = 0^3 \\ x = 0 & \text{or} & 2 - x = 0 \\ x = 0 & \text{or} & x = 2 \end{array}$$

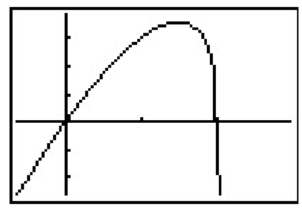
The zeros 0 and 2 divide the real number line into three test intervals. The sign diagram and accompanying graph are below. Note that the intervals on which f is $(+)$ correspond to where the graph of f is above the x-axis, and where the graph of f is below the x-axis we have that f is $(-)$. The calculator suggests something mysterious happens near $x = 2$. Zooming in shows the graph becomes nearly vertical there. You'll have to wait until Calculus to fully understand this phenomenon.

$(-)$ 0 $(+)$ 0 $(-)$

0 2



$y = f(x)$



$y = f(x)$ near $x = 2$.

2. In $g(x) = \sqrt{2 - \sqrt[4]{x+3}}$, we have two radicals both of which are even indexed. To satisfy $\sqrt[4]{x+3}$, we require $x + 3 \geq 0$ or $x \geq -3$. To satisfy $\sqrt{2 - \sqrt[4]{x+3}}$, we need $2 - \sqrt[4]{x+3} \geq 0$. While it may be tempting to write this as $2 \geq \sqrt[4]{x+3}$ and take both sides to the fourth power, there are times when this technique will produce erroneous results.[6] Instead, we solve $2 - \sqrt[4]{x+3} \geq 0$ using a sign diagram. If we let $r(x) = 2 - \sqrt[4]{x+3}$, we know $x \geq -3$, so we concern ourselves with only this portion of the number line. To find the zeros of r we set $r(x) = 0$ and solve $2 - \sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$ so that $\left(\sqrt[4]{x+3}\right)^4 = 2^4$ from which we obtain $x + 3 = 16$ or $x = 13$. Since we raised both sides of an equation to an even power, we need to check to see if $x = 13$ is an extraneous solution.[7] We find $x = 13$ does check since $2 - \sqrt[4]{x+3} = 2 - \sqrt[4]{13+3} = 2 - \sqrt[4]{16} = 2 - 2 = 0$. Below is our sign diagram for r.

$(+)$ 0 $(-)$

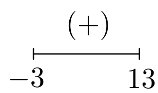
-3 13

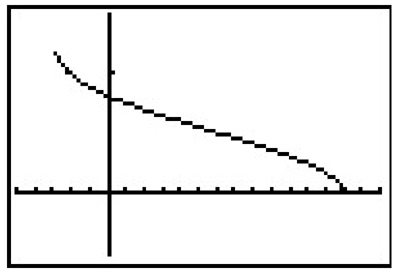
We find $2 - \sqrt[4]{x+3} \geq 0$ on $[-3, 13]$ so this is the domain of g. To find a sign diagram for g, we look for the zeros of g. Setting $g(x) = 0$ is equivalent to $\sqrt{2 - \sqrt[4]{x+3}} = 0$. After squaring

[6] For instance, $-2 \geq \sqrt[4]{x+3}$, which has no solution or $-2 \leq \sqrt[4]{x+3}$ whose solution is $[-3, \infty)$.

[7] Recall, this means we have produced a candidate which doesn't satisfy the original equation. Do you remember how raising both sides of an equation to an even power could cause this?

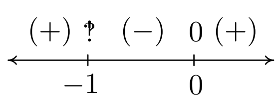
both sides, we get $2 - \sqrt[4]{x+3} = 0$, whose solution we have found to be $x = 13$. Since we squared both sides, we double check and find $g(13)$ is, in fact, 0. Our sign diagram and graph of g are below. Since the domain of g is $[-3, 13]$, what we have below is not just a *portion* of the graph of g, but the *complete* graph. It is always above or on the x-axis, which verifies our sign diagram.

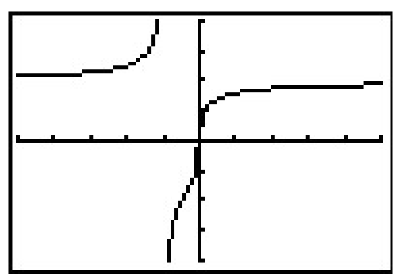




The complete graph of $y = g(x)$.

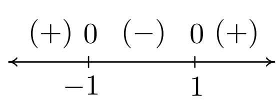
3. The radical in $h(x)$ is odd, so our only concern is the denominator. Setting $x + 1 = 0$ gives $x = -1$, so our domain is $(-\infty, -1) \cup (-1, \infty)$. To find the zeros of h, we set $h(x) = 0$. To solve $\sqrt[3]{\frac{8x}{x+1}} = 0$, we cube both sides to get $\frac{8x}{x+1} = 0$. We get $8x = 0$, or $x = 0$. Below is the resulting sign diagram and corresponding graph. From the graph, it appears as though $x = -1$ is a vertical asymptote. Carrying out an analysis as $x \to -1$ as in Section 4.2 confirms this. (We leave the details to the reader.) Near $x = 0$, we have a situation similar to $x = 2$ in the graph of f in number 1 above. Finally, it appears as if the graph of h has a horizontal asymptote $y = 2$. Using techniques from Section 4.2, we find as $x \to \pm\infty$, $\frac{8x}{x+1} \to 8$. From this, it is hardly surprising that as $x \to \pm\infty$, $h(x) = \sqrt[3]{\frac{8x}{x+1}} \approx \sqrt[3]{8} = 2$.



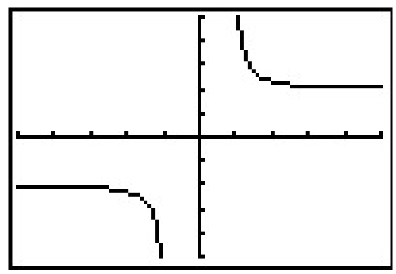


$y = h(x)$

4. To find the domain of k, we have both an even root and a denominator to concern ourselves with. To satisfy the square root, $x^2 - 1 \geq 0$. Setting $r(x) = x^2 - 1$, we find the zeros of r to be $x = \pm 1$, and we find the sign diagram of r to be



We find $x^2 - 1 \geq 0$ for $(-\infty, -1] \cup [1, \infty)$. To keep the denominator of $k(x)$ away from zero, we set $\sqrt{x^2-1} = 0$. We leave it to the reader to verify the solutions are $x = \pm 1$, both of which must be excluded from the domain. Hence, the domain of k is $(-\infty, -1) \cup (1, \infty)$. To build the sign diagram for k, we need the zeros of k. Setting $k(x) = 0$ results in $\frac{2x}{\sqrt{x^2-1}} = 0$. We get $2x = 0$ or $x = 0$. However, $x = 0$ isn't in the domain of k, which means k has no zeros. We construct our sign diagram on the domain of k below alongside the graph of k. It appears that the graph of k has two vertical asymptotes, one at $x = -1$ and one at $x = 1$. The gap in the graph between the asymptotes is because of the gap in the domain of k. Concerning end behavior, there appear to be two horizontal asymptotes, $y = 2$ and $y = -2$. To see why this is the case, we think of $x \to \pm\infty$. The radicand of the denominator $x^2 - 1 \approx x^2$, and as such, $k(x) = \frac{2x}{\sqrt{x^2-1}} \approx \frac{2x}{\sqrt{x^2}} = \frac{2x}{|x|}$. As $x \to \infty$, we have $|x| = x$ so $k(x) \approx \frac{2x}{x} = 2$. On the other hand, as $x \to -\infty$, $|x| = -x$, and as such $k(x) \approx \frac{2x}{-x} = -2$. Finally, it appears as though the graph of k passes the Horizontal Line Test which means k is one to one and k^{-1} exists. Computing k^{-1} is left as an exercise.



$y = k(x)$

□

As the previous example illustrates, the graphs of general algebraic functions can have features we've seen before, like vertical and horizontal asymptotes, but they can occur in new and exciting ways. For example, $k(x) = \frac{2x}{\sqrt{x^2-1}}$ had two distinct horizontal asymptotes. You'll recall that rational functions could have at most one horizontal asymptote. Also some new characteristics like 'unusual steepness'[8] and cusps[9] can appear in the graphs of arbitrary algebraic functions. Our next example first demonstrates how we can use sign diagrams to solve nonlinear inequalities. (Don't panic. The technique is very similar to the ones used in Chapters 2, 3 and 4.) We then check our answers graphically with a calculator and see some of the new graphical features of the functions in this extended family.

Example 5.3.2. Solve the following inequalities. Check your answers graphically with a calculator.

---

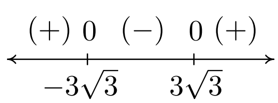
[8] The proper Calculus term for this is 'vertical tangent', but for now we'll be okay calling it 'unusual steepness'.

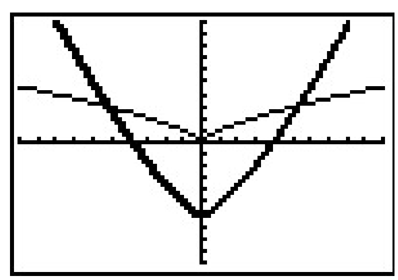
[9] See page 241 for the first reference to this feature.

1. $x^{2/3} < x^{4/3} - 6$
2. $3(2-x)^{1/3} \leq x(2-x)^{-2/3}$

Solution.

1. To solve $x^{2/3} < x^{4/3} - 6$, we get 0 on one side and attempt to solve $x^{4/3} - x^{2/3} - 6 > 0$. We set $r(x) = x^{4/3} - x^{2/3} - 6$ and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(x) = 0$ and attempt to solve $x^{4/3} - x^{2/3} - 6 = 0$. At this point, it may be unclear how to proceed. We could always try as a last resort converting back to radical notation, but in this case we can take a cue from Example 3.3.4. Since there are three terms, and the exponent on one of the variable terms, $x^{4/3}$, is exactly twice that of the other, $x^{2/3}$, we have ourselves a 'quadratic in disguise' and we can rewrite $x^{4/3} - x^{2/3} - 6 = 0$ as $\left(x^{2/3}\right)^2 - x^{2/3} - 6 = 0$. If we let $u = x^{2/3}$, then in terms of u, we get $u^2 - u - 6 = 0$. Solving for u, we obtain $u = -2$ or $u = 3$. Replacing $x^{2/3}$ back in for u, we get $x^{2/3} = -2$ or $x^{2/3} = 3$. To avoid the trouble we encountered in the discussion following Definition 5.5, we now convert back to radical notation. By interpreting $x^{2/3}$ as $\sqrt[3]{x^2}$ we have $\sqrt[3]{x^2} = -2$ or $\sqrt[3]{x^2} = 3$. Cubing both sides of these equations results in $x^2 = -8$, which admits no real solution, or $x^2 = 27$, which gives $x = \pm 3\sqrt{3}$. We construct a sign diagram and find $x^{4/3} - x^{2/3} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(x) = x^{2/3}$ and $g(x) = x^{4/3} - 6$. The solution to $x^{2/3} < x^{4/3} - 6$ corresponds to the inequality $f(x) < g(x)$, which means we are looking for the x values for which the graph of f is below the graph of g. Using the 'Intersect' command we confirm[10] that the graphs cross at $x = \pm 3\sqrt{3}$. We see that the graph of f is below the graph of g (the thicker curve) on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$.



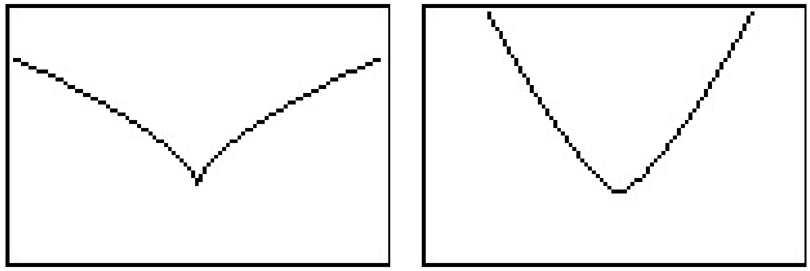


$y = f(x)$ and $\boldsymbol{y = g(x)}$

As a point of interest, if we take a closer look at the graphs of f and g near $x = 0$ with the axes off, we see that despite the fact they both involve cube roots, they exhibit different behavior near $x = 0$. The graph of f has a sharp turn, or cusp, while g does not.[11]

[10] Or at least confirm to several decimal places

[11] Again, we introduced this feature on page 241 as a feature which makes the graph of a function 'not smooth'.



$y = f(x)$ near $x = 0$ $\qquad$ $y = g(x)$ near $x = 0$

2. To solve $3(2-x)^{1/3} \leq x(2-x)^{-2/3}$, we gather all the nonzero terms on one side and obtain $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$. We set $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$. As in number 1, the denominators of the rational exponents are odd, which means there are no domain concerns there. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}}$$

Setting the denominator equal to zero we get $(2-x)^{2/3} = 0$, or $\sqrt[3]{(2-x)^2} = 0$. After cubing both sides, and subsequently taking square roots, we get $2 - x = 0$, or $x = 2$. Hence, the domain of r is $(-\infty, 2) \cup (2, \infty)$. To find the zeros of r, we set $r(x) = 0$. There are two school of thought on how to proceed and we demonstrate both.

- *Factoring Approach.* From $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since $-\frac{2}{3} < \frac{1}{3}$, we factor $(2-x)^{-2/3}$ from both quantities. While it may seem odd to do so, we need to factor $(2-x)^{-2/3}$ *from* $(2-x)^{1/3}$, which results in subtracting the exponent $-\frac{2}{3}$ from $\frac{1}{3}$. We proceed using the usual properties of exponents.[12]

$$\begin{aligned} r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\ &= (2-x)^{-2/3}\left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x)^{3/3} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x)^{1} - x\right] && \text{since } \sqrt[3]{u^3} = \left(\sqrt[3]{u}\right)^3 = u \\ &= (2-x)^{-2/3}\,(6-4x) \\ &= (2-x)^{-2/3}\,(6-4x) \end{aligned}$$

To solve $r(x) = 0$, we set $(2-x)^{-2/3}(6-4x) = 0$, or $\frac{6-4x}{(2-x)^{2/3}} = 0$. We have $6 - 4x = 0$ or $x = \frac{3}{2}$.

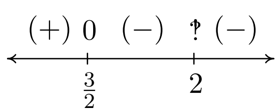
[12] And we exercise special care when reducing the $\frac{3}{3}$ power to 1.

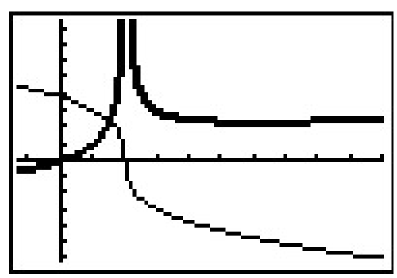
- *Common Denominator Approach.* We rewrite

$$\begin{aligned}
r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\
&= 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}} \\
&= \frac{3(2-x)^{1/3}(2-x)^{2/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{common denominator} \\
&= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
&= \frac{3(2-x)^{3/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
&= \frac{3(2-x)^{1}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{since } \sqrt[3]{u^3} = \left(\sqrt[3]{u}\right)^3 = u \\
&= \frac{3(2-x) - x}{(2-x)^{2/3}} \\
&= \frac{6-4x}{(2-x)^{2/3}}
\end{aligned}$$

  As before, when we set $r(x) = 0$ we obtain $x = \frac{3}{2}$.

We now create our sign diagram and find $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check this graphically, we set $f(x) = 3(2-x)^{1/3}$ and $g(x) = x(2-x)^{-2/3}$ (the thicker curve). We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of f is below the graph of g for $x \geq \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote.

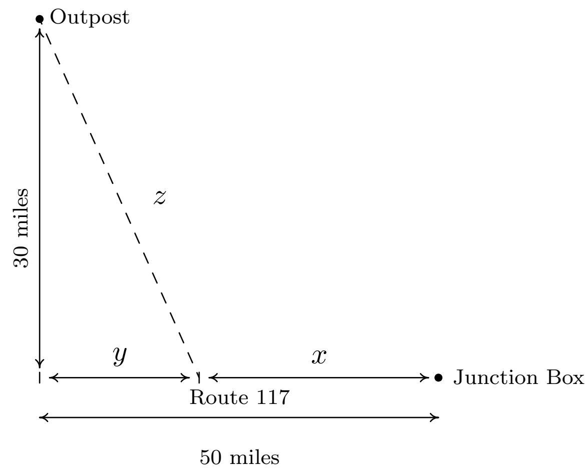




$y = f(x)$ and $\boldsymbol{y = g(x)}$ □

One application of algebraic functions was given in Example 1.6.6 in Section 1.1. Our last example is a more sophisticated application of distance.

Example 5.3.3. Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles downroad from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.



1. Express the total cost C of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.

2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?

Solution.

1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $C = 15x + 20z$. Our next goal is to determine z as a function of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Hence, $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$ so that our cost as a function of x only is given by
$$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$$
From the context of the problem, we have $0 \leq x \leq 50$.

2. Graphing $y = C(x)$ on a calculator and using the 'Minimum' feature, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be $(15.98, 1146.86)$. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live SasquatchCasts? Priceless. □

5.3.1 Exercises

For each function in Exercises 1 - 10 below

- Find its domain.
- Create a sign diagram.
- Use your calculator to help you sketch its graph and identify any vertical or horizontal asymptotes, 'unusual steepness' or cusps.

1. $f(x) = \sqrt{1 - x^2}$
2. $f(x) = \sqrt{x^2 - 1}$
3. $f(x) = x\sqrt{1 - x^2}$
4. $f(x) = x\sqrt{x^2 - 1}$
5. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
6. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
7. $f(x) = x^{\frac{2}{3}}(x - 7)^{\frac{1}{3}}$
8. $f(x) = x^{\frac{3}{2}}(x - 7)^{\frac{1}{3}}$
9. $f(x) = \sqrt{x(x + 5)(x - 4)}$
10. $f(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$

In Exercises 11 - 16, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using the transformations presented in Section 1.7.

11. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x - 1} - 2$
12. $f(x) = \sqrt[3]{x}$, $g(x) = -2\sqrt[3]{x + 1} + 4$
13. $f(x) = \sqrt[4]{x}$, $g(x) = \sqrt[4]{x - 1} - 2$
14. $f(x) = \sqrt[4]{x}$, $g(x) = 3\sqrt[4]{x - 7} - 1$
15. $f(x) = \sqrt[5]{x}$, $g(x) = \sqrt[5]{x + 2} + 3$
16. $f(x) = \sqrt[8]{x}$, $g(x) = \sqrt[8]{-x} - 2$

In Exercises 17 - 35, solve the equation or inequality.

17. $x + 1 = \sqrt{3x + 7}$
18. $2x + 1 = \sqrt{3 - 3x}$
19. $x + \sqrt{3x + 10} = -2$
20. $3x + \sqrt{6 - 9x} = 2$
21. $2x - 1 = \sqrt{x + 3}$
22. $x^{\frac{3}{2}} = 8$
23. $x^{\frac{2}{3}} = 4$
24. $\sqrt{x - 2} + \sqrt{x - 5} = 3$
25. $\sqrt{2x + 1} = 3 + \sqrt{4 - x}$
26. $5 - (4 - 2x)^{\frac{2}{3}} = 1$
27. $10 - \sqrt{x - 2} \leq 11$
28. $\sqrt[3]{x} \leq x$

29. $2(x-2)^{-\frac{1}{3}} - \frac{2}{3}x(x-2)^{-\frac{4}{3}} \leq 0$

30. $-\frac{4}{3}(x-2)^{-\frac{4}{3}} + \frac{8}{9}x(x-2)^{-\frac{7}{3}} \geq 0$

31. $2x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}} \geq 0$

32. $\sqrt[3]{x^3+3x^2-6x-8} > x+1$

33. $\frac{1}{3}x^{\frac{3}{4}}(x-3)^{-\frac{2}{3}} + \frac{3}{4}x^{-\frac{1}{4}}(x-3)^{\frac{1}{3}} < 0$

34. $x^{-\frac{1}{3}}(x-3)^{-\frac{2}{3}} - x^{-\frac{4}{3}}(x-3)^{-\frac{5}{3}}(x^2-3x+2) \geq 0$

35. $\frac{2}{3}(x+4)^{\frac{3}{5}}(x-2)^{-\frac{1}{3}} + \frac{3}{5}(x+4)^{-\frac{2}{5}}(x-2)^{\frac{2}{3}} \geq 0$

36. Rework Example 5.3.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.

37. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula $V = \frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S = \pi r\sqrt{r^2+h^2}$. Suppose a cone is to have a volume of 100 cubic centimeters.

    (a) Use the formula for volume to find the height h as a function of r.

    (b) Use the formula for surface area and your answer to 37a to find the surface area S as a function of r.

    (c) Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

38. The National Weather Service uses the following formula to calculate the wind chill:

$$W = 35.74 + 0.6215\,T_a - 35.75\,V^{0.16} + 0.4275\,T_a\,V^{0.16}$$

    where W is the wind chill temperature in °F, T_a is the air temperature in °F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.

    (a) Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.

    (b) Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Find the wind speed. Round your answer to two decimal places.

39. As a follow-up to Exercise 38, suppose the air temperature is 28°F.

    (a) Use the formula from Exercise 38 to find an expression for the wind chill temperature as a function of the wind speed, $W(V)$.

    (b) Solve $W(V) = 0$, round your answer to two decimal places, and interpret.

    (c) Graph the function W using your calculator and check your answer to part 39b.

40. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

41. The Cobb-Douglas production model states that the yearly total dollar value of the production output P in an economy is a function of labor x (the total number of hours worked in a year) and capital y (the total dollar value of all of the stuff purchased in order to make things). Specifically, $P = ax^by^{1-b}$. By fixing P, we create what's known as an 'isoquant' and we can then solve for y as a function of x. Let's assume that the Cobb-Douglas production model for the country of Sasquatchia is $P = 1.23x^{0.4}y^{0.6}$.

    (a) Let $P = 300$ and solve for y in terms of x. If $x = 100$, what is y?

    (b) Graph the isoquant $300 = 1.23x^{0.4}y^{0.6}$. What information does an ordered pair (x, y) which makes $P = 300$ give you? With the help of your classmates, find several different combinations of labor and capital all of which yield $P = 300$. Discuss any patterns you may see.

42. According to Einstein's Theory of Special Relativity, the observed mass m of an object is a function of how fast the object is traveling. Specifically,

$$m(x) = \frac{m_r}{\sqrt{1 - \frac{x^2}{c^2}}}$$

where $m(0) = m_r$ is the mass of the object at rest, x is the speed of the object and c is the speed of light.

    (a) Find the applied domain of the function.

    (b) Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.

    (c) As $x \to c^-$, what happens to $m(x)$?

    (d) How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?

43. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.

44. Suppose Fritzy the Fox, positioned at a point (x, y) in the first quadrant, spots Chewbacca the Bunny at $(0, 0)$. Chewbacca begins to run along a fence (the positive y-axis) towards his warren. Fritzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is v_1 and Fritzy's speed is v_2, the path Fritzy will take to intercept Chewbacca, provided v_2 is directly proportional to, but not equal to, v_1 is modeled by

$$y = \frac{1}{2}\left(\frac{x^{1+v_1/v_2}}{1+v_1/v_2} - \frac{x^{1-v_1/v_2}}{1-v_1/v_2}\right) + \frac{v_1 v_2}{v_2^2 - v_1^2}$$

    (a) Determine the path that Fritzy will take if he runs exactly twice as fast as Chewbacca; that is, $v_2 = 2v_1$. Use your calculator to graph this path for $x \geq 0$. What is the significance of the y-intercept of the graph?
    (b) Determine the path Fritzy will take if Chewbacca runs exactly twice as fast as he does; that is, $v_1 = 2v_2$. Use your calculator to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.
    (c) With the help of your classmates, generalize parts (a) and (b) to two cases: $v_2 > v_1$ and $v_2 < v_1$. We will discuss the case of $v_1 = v_2$ in Exercise 32 in Section 6.5.

45. Verify the Quotient Rule for Radicals in Theorem 5.6.

46. Show that $\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = x$ for all $x \geq 0$.

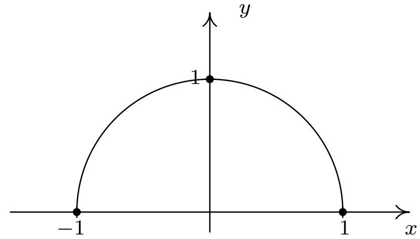
47. Show that $\sqrt[3]{2}$ is an irrational number by first showing that it is a zero of $p(x) = x^3 - 2$ and then showing p has no rational zeros. (You'll need the Rational Zeros Theorem, Theorem 3.9, in order to show this last part.)

48. With the help of your classmates, generalize Exercise 47 to show that $\sqrt[n]{c}$ is an irrational number for any natural numbers $c \geq 2$ and $n \geq 2$ provided that $c \neq p^n$ for some natural number p.

5.3.2 Answers

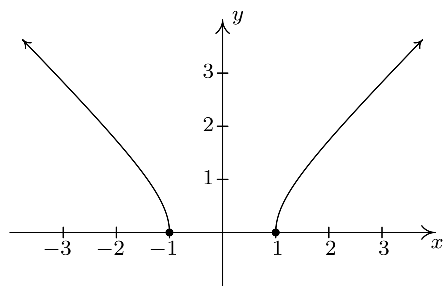
1. $f(x) = \sqrt{1-x^2}$
   Domain: $[-1, 1]$
   $0 \quad (+) \quad 0$
   $-1 \qquad\qquad 1$

   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



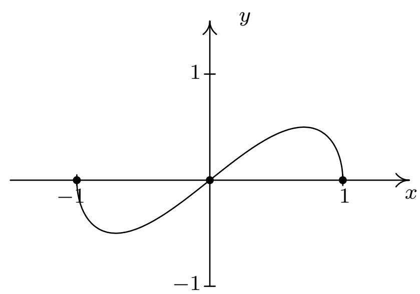
2. $f(x) = \sqrt{x^2-1}$
   Domain: $(-\infty, -1] \cup [1, \infty)$
   $(+) \quad 0 \qquad 0 \quad (+)$
   $-1 \qquad 1$

   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



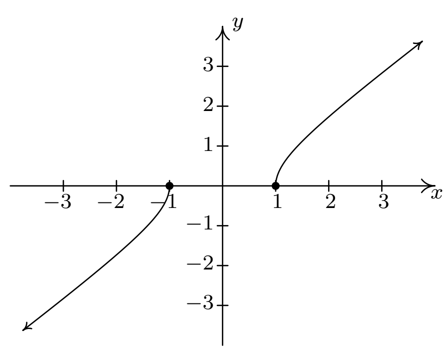
3. $f(x) = x\sqrt{1-x^2}$
   Domain: $[-1, 1]$
   $0 \quad (-) \quad 0 \quad (+) \quad 0$
   $-1 \qquad 0 \qquad 1$

   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



4. $f(x) = x\sqrt{x^2-1}$
   Domain: $(-\infty, -1] \cup [1, \infty)$
   $(-) \quad 0 \qquad 0 \quad (+)$
   $-1 \qquad 1$

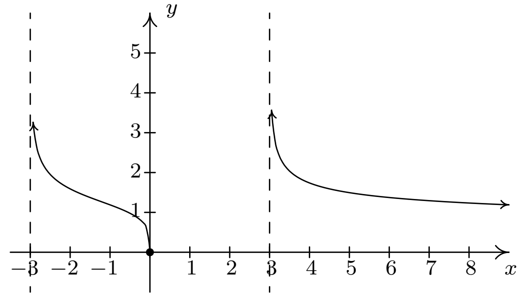
   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



5. $f(x) = \sqrt[4]{\dfrac{16x}{x^2-9}}$
Domain: $(-3, 0] \cup (3, \infty)$

‽ (+) 0 ‽ (+)
-3 0 3

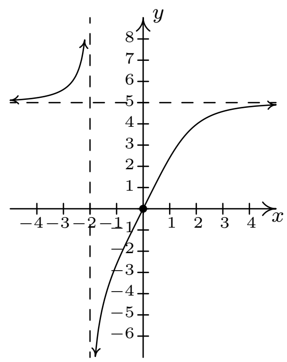
Vertical asymptotes: $x = -3$ and $x = 3$
Horizontal asymptote: $y = 0$
Unusual steepness at $x = 0$
No cusps



6. $f(x) = \dfrac{5x}{\sqrt[3]{x^3+8}}$
Domain: $(-\infty, -2) \cup (-2, \infty)$

(+) ‽ (−) 0 (+)
-2 0

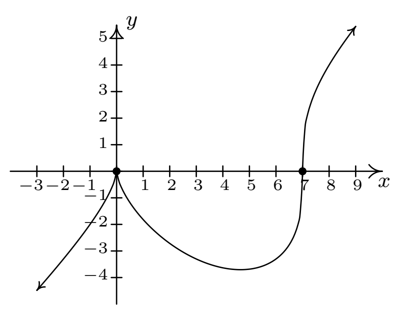
Vertical asymptote $x = -2$
Horizontal asymptote $y = 5$
No unusual steepness or cusps



7. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
Domain: $(-\infty, \infty)$

(−) 0 (−) 0 (+)
0 7

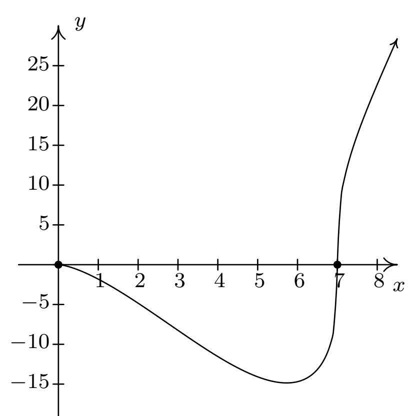
No vertical or horizontal asymptotes[13]
Unusual steepness at $x = 7$
Cusp at $x = 0$



8. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
Domain: $[0, \infty)$

0 (−) 0 (+)
0 7

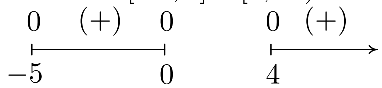
No asymptotes
Unusual steepness at $x = 7$
No cusps



[13] Using Calculus it can be shown that $y = x - \frac{7}{3}$ is a slant asymptote of this graph.

9. $f(x) = \sqrt{x(x+5)(x-4)}$

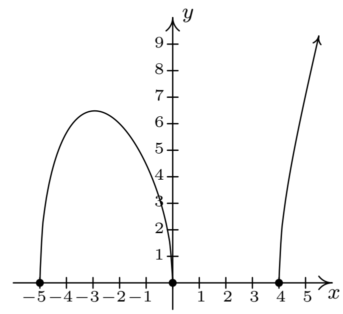
Domain: $[-5, 0] \cup [4, \infty)$



No asymptotes

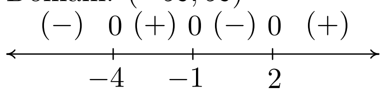
Unusual steepness at $x = -5, x = 0$ and $x = 4$

No cusps



10. $f(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$

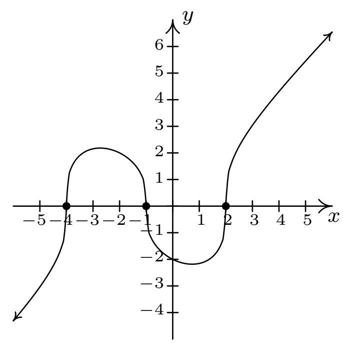
Domain: $(-\infty, \infty)$



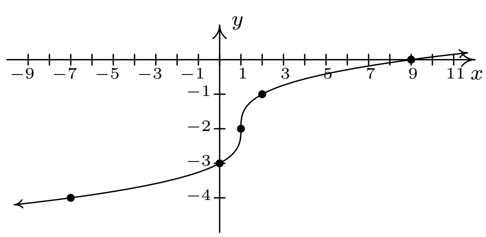
No vertical or horizontal asymptotes[14]

Unusual steepness at $x = -4, x = -1$ and $x = 2$

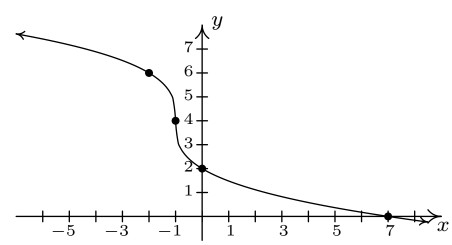
No cusps



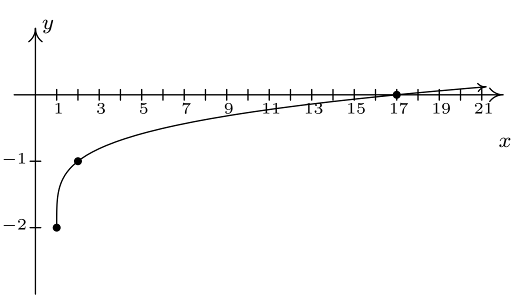
11. $g(x) = \sqrt[3]{x-1} - 2$



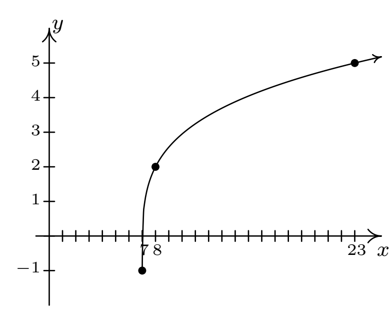
12. $g(x) = -2\sqrt[3]{x+1} + 4$



13. $g(x) = \sqrt[4]{x-1} - 2$

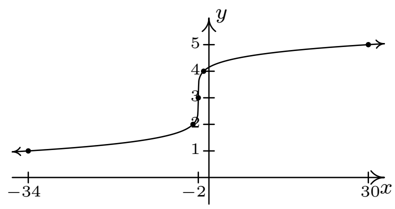


14. $g(x) = 3\sqrt[4]{x-7} - 1$

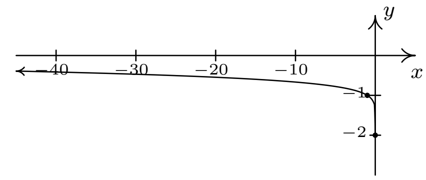


[14]Using Calculus it can be shown that $y = x + 1$ is a slant asymptote of this graph.

15. $g(x) = \sqrt[5]{x+2} + 3$



16. $g(x) = \sqrt[8]{-x} - 2$



17. $x = 3$

18. $x = \frac{1}{4}$

19. $x = -3$

20. $x = -\frac{1}{3}, \frac{2}{3}$

21. $x = \frac{5+\sqrt{57}}{8}$

22. $x = 4$

23. $x = \pm 8$

24. $x = 6$

25. $x = 4$

26. $x = -2, 6$

27. $[2, \infty)$

28. $[-1, 0] \cup [1, \infty)$

29. $(-\infty, 2) \cup (2, 3]$

30. $(2, 6]$

31. $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$

32. $(-\infty, -1)$

33. $\left(0, \frac{27}{13}\right)$

34. $(-\infty, 0) \cup (0, 3)$

35. $(-\infty, -4) \cup \left(-4, -\frac{22}{19}\right] \cup (2, \infty)$

36. $C(x) = 15x + 20\sqrt{100 + (30 - x)^2}$, $0 \leq x \leq 30$. The calculator gives the absolute minimum at $\approx (18.66, 582.29)$. This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately \$582.29.

37. (a) $h(r) = \frac{300}{\pi r^2}$, $r > 0$.

    (b) $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$

    (c) The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.

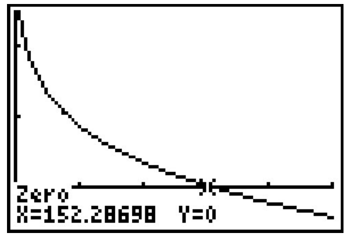
38. (a) $W \approx 37.55^{\circ}$F.

    (b) $V \approx 9.84$ miles per hour.

39. (a) $W(V) = 53.142 - 23.78V^{0.16}$. Since we are told in Exercise 38 that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to $V > 3$.

    (b) $W(V) = 0$ when $V \approx 152.29$. This means, according to the model, for the wind chill temperature to be 0°F, the wind speed needs to be 152.29 miles per hour.

(c) The graph is below.



40. $9.8\left(\frac{1}{4\pi}\right)^2 \approx 0.062$ meters or 6.2 centimeters

41. (a) First rewrite the model as $P = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$. Then $300 = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$ yields $y = \left(\frac{300}{1.23x^{\frac{2}{5}}}\right)^{\frac{5}{3}}$. If $x = 100$ then $y \approx 441.93687$.

42. (a) $[0, c)$

(b)

$$m(.1c) = \frac{m_r}{\sqrt{.99}} \approx 1.005m_r \quad m(.5c) = \frac{m_r}{\sqrt{.75}} \approx 1.155m_r$$
$$m(.9c) = \frac{m_r}{\sqrt{.19}} \approx 2.294m_r \quad m(.999c) = \frac{m_r}{\sqrt{.0.001999}} \approx 22.366m_r$$

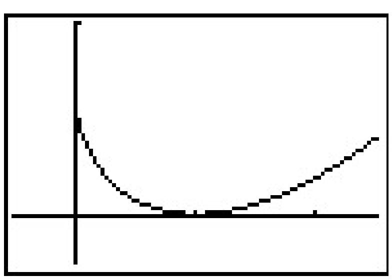
(c) As $x \to c^-$, $m(x) \to \infty$

(d) If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than $100m_r$.

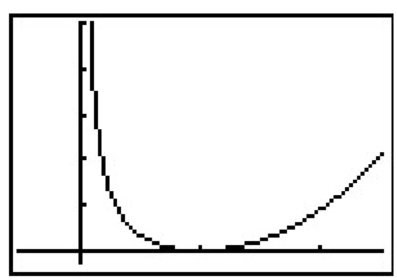
43. $k^{-1}(x) = \frac{x}{\sqrt{x^2-4}}$

44. (a) $y = \frac{1}{3}x^{3/2} - \sqrt{x} + \frac{2}{3}$. The point $\left(0, \frac{2}{3}\right)$ is when Fritzy's path crosses Chewbacca's path - in other words, where Fritzy catches Chewbacca.

(b) $y = \frac{1}{6}x^3 + \frac{1}{2x} - \frac{2}{3}$. Using the techniques from Chapter 4, we find as $x \to 0^+$, $y \to \infty$ which means, in this case, Fritzy's pursuit never ends; he never catches Chewbacca. This makes sense since Chewbacca has a head start and is running faster than Fritzy.



$y = \frac{1}{3}x^{3/2} - \sqrt{x} + \frac{2}{3}$



$y = \frac{1}{6}x^3 + \frac{1}{2x} - \frac{2}{3}$

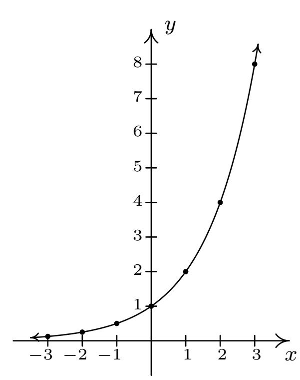
Chapter 6

Exponential and Logarithmic Functions

6.1 Introduction to Exponential and Logarithmic Functions

Of all of the functions we study in this text, exponential and logarithmic functions are possibly the ones which impact everyday life the most.[1] This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties. Up to this point, we have dealt with functions which involve terms like x^2 or $x^{2/3}$, in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with $f(x) = 2^x$. (Apparently this is a tradition. Every College Algebra book we have ever read starts with $f(x) = 2^x$.) We make a table of values, plot the points and connect the dots in a pleasing fashion.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ which we have constructed are in order. As $x \to -\infty$

[1] Take a class in Differential Equations and you'll see why.

and attains values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$,

$$2^x \approx \frac{1}{\text{very big } (+)} \approx \text{very small } (+)$$

So as $x \to -\infty$, $2^x \to 0^+$. This is represented graphically using the x-axis (the line $y = 0$) as a horizontal asymptote. On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$. As a result, our graph suggests the range of f is $(0, \infty)$. The graph of f passes the Horizontal Line Test which means f is one-to-one and hence invertible. We also note that when we 'connected the dots in a pleasing fashion', we have made the implicit assumption that $f(x) = 2^x$ is continuous[2] and has a domain of all real numbers. In particular, we have suggested that things like $2^{\sqrt{3}}$ exist as real numbers. We should take a moment to discuss what something like $2^{\sqrt{3}}$ might mean, and refer the interested reader to a solid course in Calculus for a more rigorous explanation. The number $\sqrt{3} = 1.73205\ldots$ is an irrational number[3] and as such, its decimal representation neither repeats nor terminates. We can, however, approximate $\sqrt{3}$ by terminating decimals, and it stands to reason[4] we can use these to approximate $2^{\sqrt{3}}$. For example, if we approximate $\sqrt{3}$ by 1.73, we can approximate $2^{\sqrt{3}} \approx 2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$. It is not, by any means, a pleasant number, but it is at least a number that we understand in terms of powers and roots. It also stands to reason that better and better approximations of $\sqrt{3}$ yield better and better approximations of $2^{\sqrt{3}}$, so the value of $2^{\sqrt{3}}$ should be the result of this sequence of approximations.[5]

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator, then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$. What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. We know everything we can possibly know about this function, so we exclude it from our investigations. The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ is, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 6.1. A function of the form $f(x) = b^x$ where b is a fixed real number, $b > 0$, $b \neq 1$ is called a **base *b* exponential function**.

We leave it to the reader to verify[6] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$. What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and

[2] Recall that this means there are no holes or other kinds of breaks in the graph.

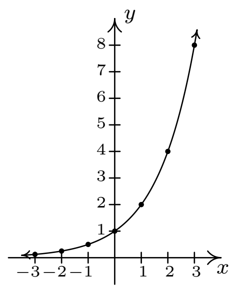
[3] You can actually prove this by considering the polynomial $p(x) = x^2 - 3$ and showing it has no rational zeros by applying Theorem 3.9.

[4] This is where Calculus and continuity come into play.

[5] Want more information? Look up "convergent sequences" on the Internet.

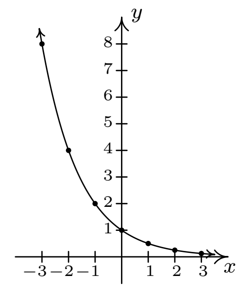
[6] Meaning, graph some more examples on your own.

note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Thinking back to Section 1.7, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis. We get



$y = f(x) = 2^x$

reflect across y-axis $\longrightarrow$ multiply each x-coordinate by -1



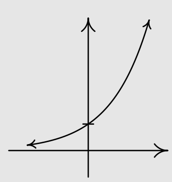
$y = g(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty$, $g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g. We summarize the basic properties of exponential functions in the following theorem.[7]

Theorem 6.1. Properties of Exponential Functions: Suppose $f(x) = b^x$.

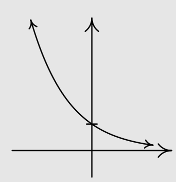
- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth[a]

- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x$, $0 < b < 1$

[a] Recall that this means the graph of f has no sharp turns or corners.

[7] The proof of which, like many things discussed in the text, requires Calculus.

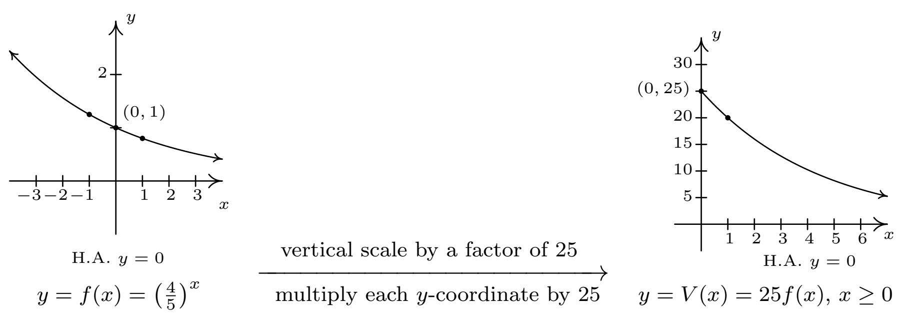
Of all of the bases for exponential functions, two occur the most often in scientific circles. The first, base 10, is often called the **common base**. The second base is an irrational number, $e \approx 2.718$, called the **natural base**. We will more formally discuss the origins of this number in Section 6.5. For now, it is enough to know that since $e > 1$, $f(x) = e^x$ is an increasing exponential function. The following examples give us an idea how these functions are used in the wild.

Example 6.1.1. The value of a car can be modeled by $V(x) = 25\left(\frac{4}{5}\right)^x$, where $x \geq 0$ is age of the car in years and $V(x)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$.
2. Sketch the graph of $y = V(x)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph you found in 2.

Solution.

1. To find $V(0)$, we replace x with 0 to obtain $V(0) = 25\left(\frac{4}{5}\right)^0 = 25$. Since x represents the age of the car in years, $x = 0$ corresponds to the car being brand new. Since $V(x)$ is measured in thousands of dollars, $V(0) = 25$ corresponds to a value of \$25,000. Putting it all together, we interpret $V(0) = 25$ to mean the purchase price of the car was \$25,000.
2. To graph $y = 25\left(\frac{4}{5}\right)^x$, we start with the basic exponential function $f(x) = \left(\frac{4}{5}\right)^x$. Since the base $b = \frac{4}{5}$ is between 0 and 1, the graph of $y = f(x)$ is decreasing. We plot the y-intercept $(0, 1)$ and two other points, $\left(-1, \frac{5}{4}\right)$ and $\left(1, \frac{4}{5}\right)$, and label the horizontal asymptote $y = 0$. To obtain $V(x) = 25\left(\frac{4}{5}\right)^x$, $x \geq 0$, we multiply the output from f by 25, in other words, $V(x) = 25f(x)$. In accordance with Theorem 1.5, this results in a vertical stretch by a factor of 25. We multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) and obtain the points $\left(-1, \frac{125}{4}\right)$, $(0, 25)$ and $(1, 20)$. The horizontal asymptote remains $y = 0$. Finally, we restrict the domain to $[0, \infty)$ to fit with the applied domain given to us. We have the result below.



3. We see from the graph of V that its horizontal asymptote is $y = 0$. (We leave it to reader to verify this analytically by thinking about what happens as we take larger and larger powers of $\frac{4}{5}$.) This means as the car gets older, its value diminishes to 0. $\square$

The function in the previous example is often called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section 6.5. For now, we present another common decay curve which will serve as the basis for further study of exponential functions. Although it may look more complicated than the previous example, it is actually just a basic exponential function which has been modified by a few transformations from Section 1.7.

Example 6.1.2. According to Newton's Law of Cooling[8] the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph.

Solution.

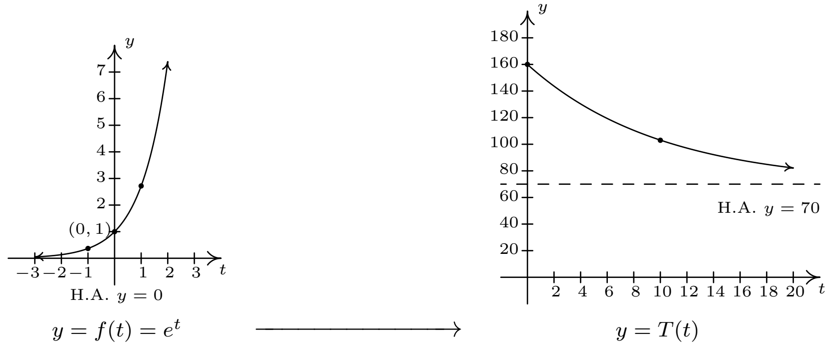
1. To find $T(0)$, we replace every occurrence of the independent variable t with 0 to obtain $T(0) = 70 + 90e^{-0.1(0)} = 160$. This means that the coffee was served at 160°F.

2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As we have already remarked, $e \approx 2.718 > 1$ so the graph of f is an increasing exponential with y-intercept $(0, 1)$ and horizontal asymptote $y = 0$. The points $\left(-1, e^{-1}\right) \approx (-1, 0.37)$ and $(1, e) \approx (1, 2.72)$ are also on the graph. Since the formula $T(t)$ looks rather complicated, we rewrite $T(t)$ in the form presented in Theorem 1.7 and use that result to track the changes to our three points and the horizontal asymptote. We have

$$T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$$

   Multiplication of the input to f, t, by -0.1 results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis. We divide each of the x values of our points by -0.1 (which amounts to multiplying them by -10) to obtain $\left(10, e^{-1}\right)$, $(0, 1)$, and $(-10, e)$. Since none of these changes affected the y values, the horizontal asymptote remains $y = 0$. Next, we see that the output from f is being multiplied by 90. This results in a vertical stretch by a factor of 90. We multiply the y-coordinates by 90 to obtain $\left(10, 90e^{-1}\right)$, $(0, 90)$, and $(-10, 90e)$. We also multiply the y value of the horizontal asymptote $y = 0$ by 90, and it remains $y = 0$. Finally, we add 70 to all of the y-coordinates, which shifts the graph upwards to obtain $\left(10, 90e^{-1} + 70\right) \approx (10, 103.11)$, $(0, 160)$, and $(-10, 90e + 70) \approx (-10, 314.64)$. Adding 70 to the horizontal asymptote shifts it upwards as well to $y = 70$. We connect these three points using the same shape in the same direction as in the graph of f and, last but not least, we restrict the domain to match the applied domain $[0, \infty)$. The result is below.

[8] We will discuss this in greater detail in Section 6.5.



3. From the graph, we see that the horizontal asymptote is $y = 70$. It is worth a moment or two of our time to see how this happens analytically and to review some of the 'number sense' developed in Chapter 4. As $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$,

$$e^{\text{very big }(-)} = \frac{1}{e^{\text{very big }(+)}} \approx \frac{1}{\text{very big }(+)} \approx \text{very small }(+)$$

The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small $(+)$. Hence, $T(t) \approx 70 +$ very small $(+)$ which means the graph is approaching the horizontal line $y = 70$ from above. This means that as time goes by, the temperature of the coffee is cooling to 70°F, presumably room temperature. □

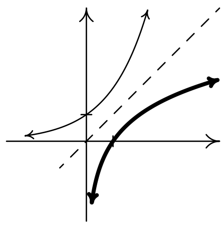
As we have already remarked, the graphs of $f(x) = b^x$ all pass the Horizontal Line Test. Thus the exponential functions are invertible. We now turn our attention to these inverses, the logarithmic functions, which are called 'logs' for short.

Definition 6.2. The inverse of the exponential function $f(x) = b^x$ is called the **base *b* logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$ We read '$\log_b(x)$' as 'log base b of x.'

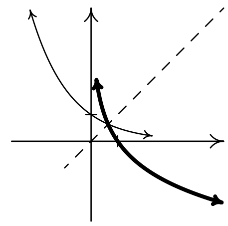
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 6.3. The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$. The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

Since logs are defined as the inverses of exponential functions, we can use Theorems 5.2 and 5.3 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$. Since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$ as shown below. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$
$\boldsymbol{y = \log_b(x)}$, $b > 1$



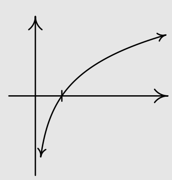
$y = b^x$, $0 < b < 1$
$\boldsymbol{y = \log_b(x)}$, $0 < b < 1$

On a procedural level, logs undo the exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 6.2. Properties of Logarithmic Functions: Suppose $f(x) = \log_b(x)$.

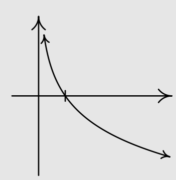
- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 6.2 is a consequence of Theorems 5.2 and 5.3. However, it is worth the reader's time to understand Theorem 6.2 from an exponential perspective. For instance, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't. For example, if we try to find $\log_2(-1)$, we are trying to find the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive. While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 6.2 at this level instead of just merely memorizing the facts.

Example 6.1.3. Simplify the following.

1. $\log_3(81)$
2. $\log_2\left(\dfrac{1}{8}\right)$
3. $\log_{\sqrt{5}}(25)$
4. $\ln\left(\sqrt[3]{e^2}\right)$
5. $\log(0.001)$
6. $2^{\log_2(8)}$
7. $117^{-\log_{117}(6)}$

Solution.

1. The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.
2. To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.
3. To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.
4. First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.
5. Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.
6. We can use Theorem 6.2 directly to simplify $2^{\log_2(8)} = 8$. We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$. We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.
7. From Theorem 6.2, we know $117^{\log_{117}(6)} = 6$, but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$. (Can you see why?) At this point, we use a property of exponents followed by Theorem 6.2 to get[9]

$$117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$$

$\square$

[9]It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117. By definition we get 6. In other words, the exponential function $f(x) = 117^x$ undoes the logarithmic function $g(x) = \log_{117}(x)$.

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument[10] of the log must be strictly positive.

Example 6.1.4. Find the domain of the following functions. Check your answers graphically using the calculator.

1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Solution.

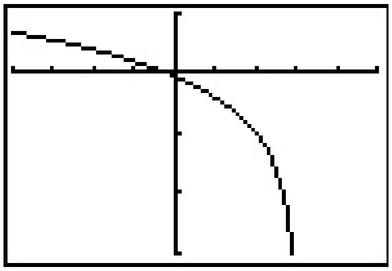
1. We set $3-x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. The graph from the calculator below verifies this. Note that we could have graphed f using transformations. Taking a cue from Theorem 1.7, we rewrite $f(x) = 2\log_{10}(-x+3)-1$ and find the main function involved is $y = h(x) = \log_{10}(x)$. We select three points to track, $\left(\frac{1}{10}, -1\right)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$. Since $f(x) = 2h(-x + 3) - 1$, Theorem 1.7 tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis). These transformations apply to the vertical asymptote $x = 0$ as well. Subtracting 3 gives us $x = -3$ as our asymptote, then multplying by -1 gives us the vertical asymptote $x = 3$. Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit. We leave it to the reader to perform the indicated arithmetic on the points themselves and to verify the graph produced by the calculator below.

2. To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$. As usual, we proceed using a sign diagram. If we define $r(x) = \frac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram below.

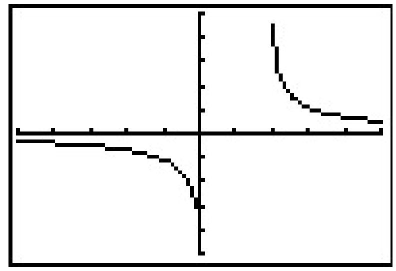
   $$\begin{array}{ccccc} (+) & 0 & (-) & ‽ & (+) \\ \longleftarrow & + & \text{---} & + & \longrightarrow \\ & 0 & & 1 & \end{array}$$

   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ to get the domain of g. The graph of $y = g(x)$ confirms this. We can tell from the graph of g that it is not the result of Section 1.7 transformations being applied to the graph $y = \ln(x)$, so barring a more detailed analysis using Calculus, the calculator graph is the best we can do. One thing worthy of note, however, is the end behavior of g. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. Using results from Chapter 4 and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$.

[10] See page 55 if you've forgotten what this term means.



$y = f(x) = 2\log(3-x) - 1$



$y = g(x) = \ln\left(\dfrac{x}{x-1}\right)$ □

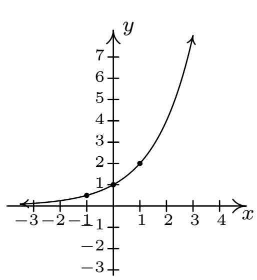
While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example solidifies this and reviews all of the material in the section.

Example 6.1.5. Let $f(x) = 2^{x-1} - 3$.

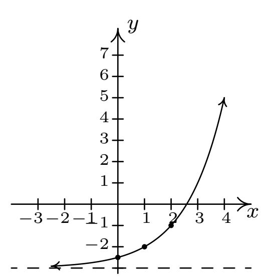
1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check the symmetry about the line $y = x$.

Solution.

1. If we identify $g(x) = 2^x$, we see $f(x) = g(x-1) - 3$. We pick the points $\left(-1, \frac{1}{2}\right)$, $(0,1)$ and $(1,2)$ on the graph of g along with the horizontal asymptote $y = 0$ to track through the transformations. By Theorem 1.7 we first add 1 to the x-coordinates of the points on the graph of g (shifting g to the right 1 unit) to get $\left(0, \frac{1}{2}\right)$, $(1,1)$ and $(2,2)$. The horizontal asymptote remains $y = 0$. Next, we subtract 3 from the y-coordinates, shifting the graph down 3 units. We get the points $\left(0, -\frac{5}{2}\right)$, $(1,-2)$ and $(2,-1)$ with the horizontal asymptote now at $y = -3$. Connecting the dots in the order and manner as they were on the graph of g, we get the graph below. We see that the domain of f is the same as g, namely $(-\infty, \infty)$, but that the range of f is $(-3, \infty)$.



$y = h(x) = 2^x$ $\longrightarrow$ $y = f(x) = 2^{x-1} - 3$



2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible. To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} from a procedural perspective. If we break $f(x) = 2^{x-1} - 3$ into a series of steps, we find f takes an input x and applies the steps

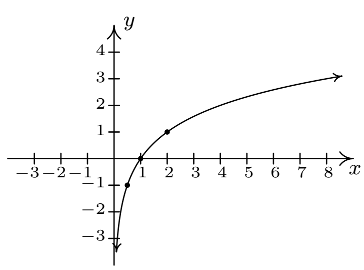
   (a) subtract 1
   (b) put as an exponent on 2
   (c) subtract 3

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm. By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should

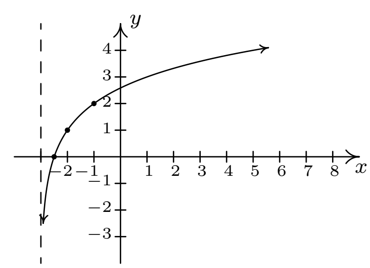
   (a) add 3
   (b) take the logarithm base 2
   (c) add 1

   In symbols, $f^{-1}(x) = \log_2(x + 3) + 1$.

3. To graph $f^{-1}(x) = \log_2(x + 3) + 1$ using transformations, we start with $j(x) = \log_2(x)$. We track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations using Theorem 1.7. Since $f^{-1}(x) = j(x + 3) + 1$, we first subtract 3 from each of the x values (including the vertical asymptote) to obtain $\left(-\frac{5}{2}, -1\right)$, $(-2, 0)$ and $(-1, 1)$ with a vertical asymptote $x = -3$. Next, we add 1 to the y values on the graph and get $\left(-\frac{5}{2}, 0\right)$, $(-2, 1)$ and $(-1, 2)$. If you are experiencing *déjà vu*, there is a good reason for it but we leave it to the reader to determine the source of this uncanny familiarity. We obtain the graph below. The domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f.



$y = j(x) = \log_2(x)$ $\longrightarrow$



$y = f^{-1}(x) = \log_2(x + 3) + 1$

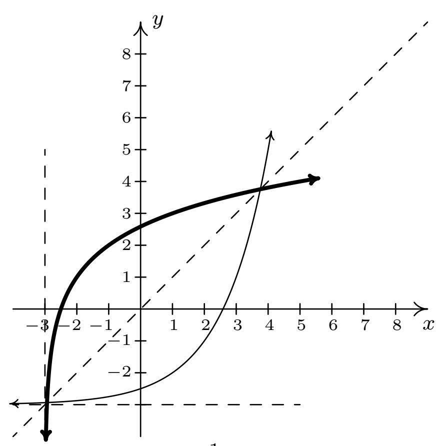
4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x + 3) + 1$ satisfy the composition requirement for inverses. For all real numbers x,

$$
\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(2^{x-1}-3\right) \\
&= \log_2\left(\left[2^{x-1}-3\right]+3\right)+1 \\
&= \log_2\left(2^{x-1}\right)+1 \\
&= (x-1)+1 && \text{Since } \log_2\left(2^u\right)=u \text{ for all real numbers } u \\
&= x \checkmark
\end{aligned}
$$

For all real numbers $x > -3$, we have[11]

$$
\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\log_2(x+3)+1\right) \\
&= 2^{(\log_2(x+3)+1)-1}-3 \\
&= 2^{\log_2(x+3)}-3 \\
&= (x+3)-3 && \text{Since } 2^{\log_2(u)}=u \text{ for all real numbers } u>0 \\
&= x \checkmark
\end{aligned}
$$

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and see the symmetry about the line $y = x$.



$y = f(x) = 2^{x-1} - 3$

$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

□

[11] Pay attention - can you spot in which step below we need $x > -3$?

6.1.1 Exercises

In Exercises 1 - 15, use the property: $b^a = c$ if and only if $\log_b(c) = a$ from Theorem 6.2 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3 = 8$
2. $5^{-3} = \frac{1}{125}$
3. $4^{5/2} = 32$
4. $\left(\frac{1}{3}\right)^{-2} = 9$
5. $\left(\frac{4}{25}\right)^{-1/2} = \frac{5}{2}$
6. $10^{-3} = 0.001$
7. $e^0 = 1$
8. $\log_5(25) = 2$
9. $\log_{25}(5) = \frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right) = -4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$
12. $\log(100) = 2$
13. $\log(0.1) = -1$
14. $\ln(e) = 1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

In Exercises 16 - 42, evaluate the expression.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}\left(\sqrt{13}\right)$
31. $\log_{36}\left(\sqrt[4]{36}\right)$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$
37. $\log\left(\sqrt[3]{10^5}\right)$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)$
39. $\log_5\left(3^{\log_3(5)}\right)$
40. $\log\left(e^{\ln(100)}\right)$
41. $\log_2\left(3^{-\log_3(2)}\right)$
42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 - 57, find the domain of the function.

43. $f(x) = \ln(x^2 + 1)$
44. $f(x) = \log_7(4x + 8)$
45. $f(x) = \ln(4x - 20)$
46. $f(x) = \log\left(x^2 + 9x + 18\right)$

47. $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$

48. $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$

49. $f(x) = \ln(7-x) + \ln(x-4)$

50. $f(x) = \ln(4x-20) + \ln\left(x^2+9x+18\right)$

51. $f(x) = \log\left(x^2+x+1\right)$

52. $f(x) = \sqrt[4]{\log_4(x)}$

53. $f(x) = \log_9(|x+3|-4)$

54. $f(x) = \ln(\sqrt{x-4}-3)$

55. $f(x) = \dfrac{1}{3-\log_5(x)}$

56. $f(x) = \dfrac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$

57. $f(x) = \ln(-2x^3-x^2+13x-6)$

In Exercises 58 - 63, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

58. $f(x) = 2^x$, $g(x) = 2^x - 1$

59. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$

60. $f(x) = 3^x$, $g(x) = 3^{-x} + 2$

61. $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$

62. $f(x) = e^x$, $g(x) = 8 - e^{-x}$

63. $f(x) = e^x$, $g(x) = 10e^{-0.1x}$

In Exercises 64 - 69, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

64. $f(x) = \log_2(x)$, $g(x) = \log_2(x+1)$

65. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{3}}(x) + 1$

66. $f(x) = \log_3(x)$, $g(x) = -\log_3(x-2)$

67. $f(x) = \log(x)$, $g(x) = 2\log(x+20) - 1$

68. $f(x) = \ln(x)$, $g(x) = -\ln(8-x)$

69. $f(x) = \ln(x)$, $g(x) = -10\ln\left(\frac{x}{10}\right)$

70. Verify that each function in Exercises 64 - 69 is the inverse of the corresponding function in Exercises 58 - 63. (Match up #58 and #64, and so on.)

In Exercises 71 - 74, find the inverse of the function from the 'procedural perspective' discussed in Example 6.1.5 and graph the function and its inverse on the same set of axes.

71. $f(x) = 3^{x+2} - 4$

72. $f(x) = \log_4(x-1)$

73. $f(x) = -2^{-x} + 1$

74. $f(x) = 5\log(x) - 2$

(Logarithmic Scales) In Exercises 75 - 77, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

75. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[12] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event", which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

   (a) Show that $M(0.001) = 0$.
   (b) Compute $M(80,000)$.
   (c) Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.
   (d) Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

76. While the decibel scale can be used in many disciplines,[13] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound.[14] The Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\frac{W}{m^2}$ is the threshold of human hearing.

   (a) Compute $L(10^{-6})$.

[12] Rock-solid, perhaps?

[13] See this webpage for more information.

[14] As of the writing of this exercise, the Wikipedia page given here states that it may not meet the "general notability guideline" nor does it cite any references or sources. I find this odd because it is this very usage of the decibel scale which shows up in every College Algebra book I have read. Perhaps those other books have been wrong all along and we're just blindly following tradition.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

77. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

    (a) The hydrogen ion concentration of pure water is $[\text{H}^+] = 10^{-7}$. Find its pH.

    (b) Find the pH of a solution with $[\text{H}^+] = 6.3 \times 10^{-13}$.

    (c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

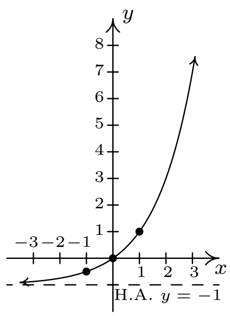
78. Show that $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0,\ b \neq 1$.

79. (Crazy bonus question) Without using your calculator, determine which is larger: e^π or π^e.

6.1.2 Answers

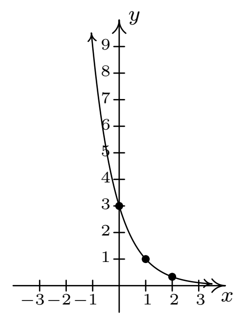
1. $\log_2(8) = 3$
2. $\log_5\left(\frac{1}{125}\right) = -3$
3. $\log_4(32) = \frac{5}{2}$
4. $\log_{\frac{1}{3}}(9) = -2$
5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$
6. $\log(0.001) = -3$
7. $\ln(1) = 0$
8. $5^2 = 25$
9. $(25)^{\frac{1}{2}} = 5$
10. $3^{-4} = \frac{1}{81}$
11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$
12. $10^2 = 100$
13. $10^{-1} = 0.1$
14. $e^1 = e$
15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$
16. $\log_3(27) = 3$
17. $\log_6(216) = 3$
18. $\log_2(32) = 5$
19. $\log_6\left(\frac{1}{36}\right) = -2$
20. $\log_8(4) = \frac{2}{3}$
21. $\log_{36}(216) = \frac{3}{2}$
22. $\log_{\frac{1}{5}}(625) = -4$
23. $\log_{\frac{1}{6}}(216) = -3$
24. $\log_{36}(36) = 1$
25. $\log \frac{1}{1000000} = -6$
26. $\log(0.01) = -2$
27. $\ln\left(e^3\right) = 3$
28. $\log_4(8) = \frac{3}{2}$
29. $\log_6(1) = 0$
30. $\log_{13}\left(\sqrt{13}\right) = \frac{1}{2}$
31. $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
32. $7^{\log_7(3)} = 3$
33. $36^{\log_{36}(216)} = 216$
34. $\log_{36}\left(36^{216}\right) = 216$
35. $\ln(e^5) = 5$
36. $\log\left(\sqrt[9]{10^{11}}\right) = \frac{11}{9}$
37. $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$
38. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$
39. $\log_5\left(3^{\log_3 5}\right) = 1$
40. $\log\left(e^{\ln(100)}\right) = 2$
41. $\log_2\left(3^{-\log_3(2)}\right) = -1$
42. $\ln\left(42^{6\log(1)}\right) = 0$
43. $(-\infty, \infty)$
44. $(-2, \infty)$
45. $(5, \infty)$
46. $(-\infty, -6) \cup (-3, \infty)$
47. $(-2, -1) \cup (1, \infty)$
48. $(-6, -3) \cup (5, \infty)$
49. $(4, 7)$
50. $(5, \infty)$
51. $(-\infty, \infty)$
52. $[1, \infty)$
53. $(-\infty, -7) \cup (1, \infty)$
54. $(13, \infty)$
55. $(0, 125) \cup (125, \infty)$
56. No domain
57. $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

58. Domain of g: $(-\infty, \infty)$
Range of g: $(-1, \infty)$



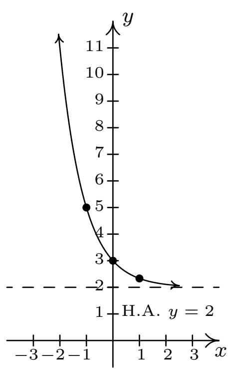
$y = g(x) = 2^x - 1$

59. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$



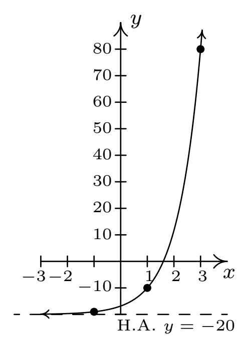
$y = g(x) = \left(\frac{1}{3}\right)^{x-1}$

60. Domain of g: $(-\infty, \infty)$
Range of g: $(2, \infty)$



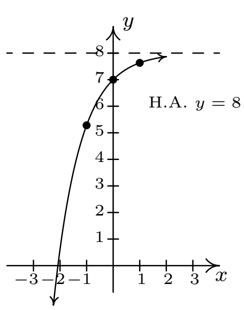
$y = g(x) = 3^{-x} + 2$

61. Domain of g: $(-\infty, \infty)$
Range of g: $(-20, \infty)$



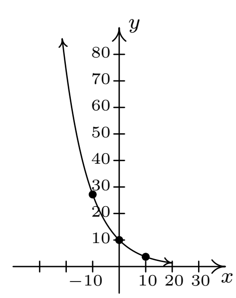
$y = g(x) = 10^{\frac{x+1}{2}} - 20$

62. Domain of g: $(-\infty, \infty)$
Range of g: $(-\infty, 8)$



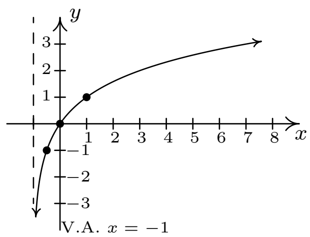
$y = g(x) = 8 - e^{-x}$

63. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$



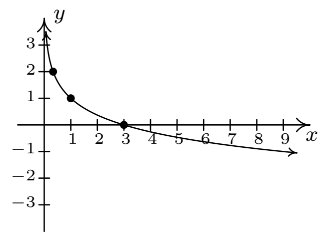
$y = g(x) = 10e^{-0.1x}$

64. Domain of g: $(-1, \infty)$
Range of g: $(-\infty, \infty)$



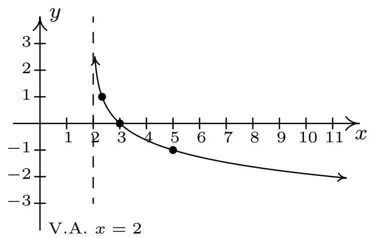
$y = g(x) = \log_2(x + 1)$

65. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$



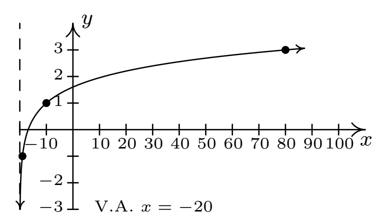
$y = g(x) = \log_{\frac{1}{3}}(x) + 1$

66. Domain of g: $(2, \infty)$
Range of g: $(-\infty, \infty)$



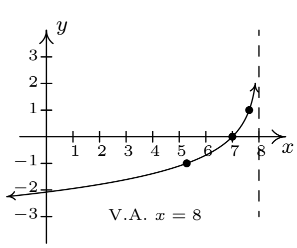
$y = g(x) = -\log_3(x - 2)$

67. Domain of g: $(-20, \infty)$
Range of g: $(-\infty, \infty)$



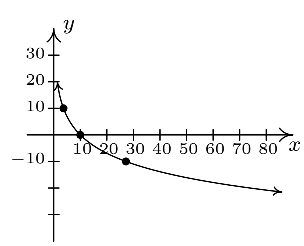
$y = g(x) = 2\log(x + 20) - 1$

68. Domain of g: $(-\infty, 8)$
Range of g:$(-\infty, \infty)$



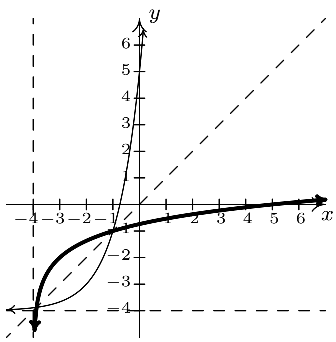
$y = g(x) = -\ln(8 - x)$

69. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$



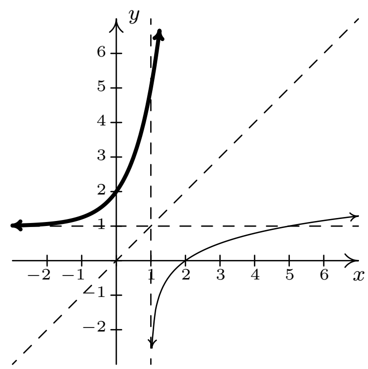
$y = g(x) = -10\ln\left(\frac{x}{10}\right)$

71. $f(x) = 3^{x+2} - 4$
$f^{-1}(x) = \log_3(x+4) - 2$



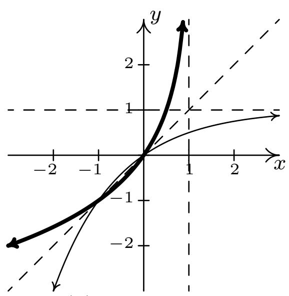
$y = f(x) = 3^{x+2} - 4$
$\boldsymbol{y = f^{-1}(x) = \log_3(x+4) - 2}$

72. $f(x) = \log_4(x-1)$
$f^{-1}(x) = 4^x + 1$



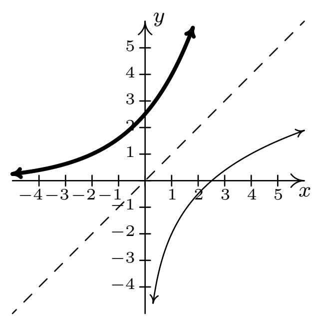
$y = f(x) = \log_4(x-1)$
$\boldsymbol{y = f^{-1}(x) = 4^x + 1}$

73. $f(x) = -2^{-x} + 1$
$f^{-1}(x) = -\log_2(1-x)$



$y = f(x) = -2^{-x} + 1$
$\boldsymbol{y = f^{-1}(x) = -\log_2(1-x)}$

74. $f(x) = 5\log(x) - 2$
$f^{-1}(x) = 10^{\frac{x+2}{5}}$



$y = f(x) = 5\log(x) - 2$
$\boldsymbol{y = f^{-1}(x) = 10^{\frac{x+2}{5}}}$

75. (a) $M(0.001) = \log\left(\frac{0.001}{0.001}\right) = \log(1) = 0.$
(b) $M(80,000) = \log\left(\frac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9.$

76. (a) $L(10^{-6}) = 60$ decibels.
(b) $I = 10^{-.5} \approx 0.316$ watts per square meter.
(c) Since $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

77. (a) The pH of pure water is 7.
(b) If $[\mathrm{H}^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.
(c) $[\mathrm{H}^+] = 10^{-0.7} \approx .1995$ moles per liter.

6.2 Properties of Logarithms

In Section 6.1, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing. As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 6.2 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 6.3. (Inverse Properties of Exponential and Logarithmic Functions) Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 6.4. (One-to-one Properties of Exponential and Logarithmic Functions) Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

We now state the algebraic properties of exponential functions which will serve as a basis for the properties of logarithms. While these properties may look identical to the ones you learned in Elementary and Intermediate Algebra, they apply to real number exponents, not just rational exponents. Note that in the theorem that follows, we are interested in the properties of exponential functions, so the base b is restricted to $b > 0$, $b \neq 1$. An added benefit of this restriction is that it eliminates the pathologies discussed in Section 5.3 when, for example, we simplified $\left(x^{2/3}\right)^{3/2}$ and obtained $|x|$ instead of what we had expected from the arithmetic in the exponents, $x^1 = x$.

Theorem 6.5. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

While the properties listed in Theorem 6.5 are certainly believable based on similar properties of integer and rational exponents, the full proofs require Calculus. To each of these properties of exponential functions corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 6.6. (Algebraic Properties of Logarithmic Functions) Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 6.6 is true. Consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$. Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$. By the one-to-one property of b^x, we have $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly. From a purely functional approach, we can see the properties in Theorem 6.6 as an example of how inverse functions interchange the roles of inputs in outputs. For instance, the Product Rule for exponential functions given in Theorem 6.5, $f(u+w) = f(u)f(w)$, says that adding inputs results in multiplying outputs. Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. Since the outputs from f are the inputs to f^{-1} and vice-versa, we have that that f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely what the Product Rule for Logarithmic functions states in Theorem 6.6: $g(uw) = g(u)+g(w)$. The reader is encouraged to view the remaining properties listed in Theorem 6.6 similarly. The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 6.6 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log.[1]

Example 6.2.1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\dfrac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\dfrac{3}{ex}\right)^2$
4. $\log\sqrt[3]{\dfrac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2 - 4\right)$

Solution.

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u = 8$ and $w = x$ and simplify.

[1] Interestingly enough, it is the exact *opposite* process (which we will practice later) that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

$$\begin{aligned}
\log_2\left(\frac{8}{x}\right) &= \log_2(8) - \log_2(x) && \text{Quotient Rule} \\
&= 3 - \log_2(x) && \text{Since } 2^3 = 8 \\
&= -\log_2(x) + 3
\end{aligned}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product. In order to use the Product Rule, the *entire* quantity inside the logarithm must be raised to the same exponent. Since the exponent 2 applies only to the x, we first apply the Product Rule with $u = 10$ and $w = x^2$. Once we get the x^2 by itself inside the log, we may apply the Power Rule with $u = x$ and $w = 2$ and simplify.

$$\begin{aligned}
\log_{0.1}\left(10x^2\right) &= \log_{0.1}(10) + \log_{0.1}\left(x^2\right) && \text{Product Rule} \\
&= \log_{0.1}(10) + 2\log_{0.1}(x) && \text{Power Rule} \\
&= -1 + 2\log_{0.1}(x) && \text{Since } (0.1)^{-1} = 10 \\
&= 2\log_{0.1}(x) - 1
\end{aligned}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{ex}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{ex}$ and $w = 2$. Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = ex$, so we replace $\ln\left(\frac{3}{ex}\right)$ with the quantity $\ln(3) - \ln(ex)$. Since $\ln\left(\frac{3}{ex}\right)$ is being multiplied by 2, the entire quantity $\ln(3) - \ln(ex)$ is multiplied by 2. Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(ex)$ with the quantity $\ln(e) + \ln(x)$, and simplify, keeping in mind that the natural log is log base e.

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Rule} \\
&= 2\left[\ln(3) - \ln(ex)\right] && \text{Quotient Rule} \\
&= 2\ln(3) - 2\ln(ex) \\
&= 2\ln(3) - 2\left[\ln(e) + \ln(x)\right] && \text{Product Rule} \\
&= 2\ln(3) - 2\ln(e) - 2\ln(x) \\
&= 2\ln(3) - 2 - 2\ln(x) && \text{Since } e^1 = e \\
&= -2\ln(x) + 2\ln(3) - 2
\end{aligned}$$

4. In Theorem 6.6, there is no mention of how to deal with radicals. However, thinking back to Definition 5.5, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power

Rule[2], and we keep in mind that the common log is log base 10.

$$\begin{array}{rcll}
\log \sqrt[3]{\dfrac{100x^2}{yz^5}} & = & \log \left(\dfrac{100x^2}{yz^5}\right)^{1/3} & \\
& = & \frac{1}{3}\log\left(\dfrac{100x^2}{yz^5}\right) & \text{Power Rule} \\
& = & \frac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] & \text{Quotient Rule} \\
& = & \frac{1}{3}\log\left(100x^2\right) - \frac{1}{3}\log\left(yz^5\right) & \\
& = & \frac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \frac{1}{3}\left[\log(y) + \log\left(z^5\right)\right] & \text{Product Rule} \\
& = & \frac{1}{3}\log(100) + \frac{1}{3}\log\left(x^2\right) - \frac{1}{3}\log(y) - \frac{1}{3}\log\left(z^5\right) & \\
& = & \frac{1}{3}\log(100) + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) & \text{Power Rule} \\
& = & \frac{2}{3} + \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) & \text{Since } 10^2 = 100 \\
& = & \frac{2}{3}\log(x) - \frac{1}{3}\log(y) - \frac{5}{3}\log(z) + \frac{2}{3} &
\end{array}$$

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2-4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2-4=(x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule.

$$\begin{array}{rcll}
\log_{117}\left(x^2-4\right) & = & \log_{117}\left[(x+2)(x-2)\right] & \text{Factor} \\
& = & \log_{117}(x+2) + \log_{117}(x-2) & \text{Product Rule}
\end{array}$$

□

A couple of remarks about Example 6.2.1 are in order. First, while not explicitly stated in the above example, a general rule of thumb to determine which log property to apply first to a complicated problem is 'reverse order of operations.' For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. In a multi-step problem, this rule can give the required guidance on which log property to apply at each step. The reader is encouraged to look through the solutions to Example 6.2.1 to see this rule in action. Second, while we were instructed to assume when necessary that all quantities represented positive real numbers, the authors would be committing a sin of omission if we failed to point out that, for instance, the functions $f(x)=\log_{117}\left(x^2-4\right)$ and $g(x)=\log_{117}(x+2)+\log_{117}(x-2)$ have different domains, and, hence, are different functions. We leave it to the reader to verify the domain of f is $(-\infty,-2)\cup(2,\infty)$ whereas the domain of g is $(2,\infty)$. In general, when using log properties to

[2] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

expand a logarithm, we may very well be restricting the domain as we do so. One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right) = \log_{117}\left(x^2\right) - \log_{117}(4)$, which simply isn't true, in general. The unwritten[3] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 6.2.2. Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1) - \log_3(x+1)$
2. $\log(x) + 2\log(y) - \log(z)$
3. $4\log_2(x) + 3$
4. $-\ln(x) - \frac{1}{2}$

Solution. Whereas in Example 6.2.1 we read the properties in Theorem 6.6 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1) - \log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$.

2. In the expression, $\log(x) + 2\log(y) - \log(z)$, we have both a sum and difference of logarithms. However, before we use the product rule to combine $\log(x) + 2\log(y)$, we note that we need to somehow deal with the coefficient 2 on $\log(y)$. This can be handled using the Power Rule. We can then apply the Product and Quotient Rules as we move from left to right. Putting it all together, we have

$$\begin{aligned} \log(x) + 2\log(y) - \log(z) &= \log(x) + \log\left(y^2\right) - \log(z) && \text{Power Rule} \\ &= \log\left(xy^2\right) - \log(z) && \text{Product Rule} \\ &= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule} \end{aligned}$$

3. We can certainly get started rewriting $4\log_2(x) + 3$ by applying the Power Rule to $4\log_2(x)$ to obtain $\log_2\left(x^4\right)$, but in order to use the Product Rule to handle the addition, we need to rewrite 3 as a logarithm base 2. From Theorem 6.3, we know $3 = \log_2\left(2^3\right)$, so we get

$$\begin{aligned} 4\log_2(x) + 3 &= \log_2\left(x^4\right) + 3 && \text{Power Rule} \\ &= \log_2\left(x^4\right) + \log_2\left(2^3\right) && \text{Since } 3 = \log_2\left(2^3\right) \\ &= \log_2\left(x^4\right) + \log_2(8) \\ &= \log_2\left(8x^4\right) && \text{Product Rule} \end{aligned}$$

[3]The authors relish the irony involved in writing what follows.

4. To get started with $-\ln(x) - \frac{1}{2}$, we rewrite $-\ln(x)$ as $(-1)\ln(x)$. We can then use the Power Rule to obtain $(-1)\ln(x) = \ln\left(x^{-1}\right)$. In order to use the Quotient Rule, we need to write $\frac{1}{2}$ as a natural logarithm. Theorem 6.3 gives us $\frac{1}{2} = \ln\left(e^{1/2}\right) = \ln\left(\sqrt{e}\right)$. We have

$$\begin{array}{rcll} -\ln(x) - \frac{1}{2} & = & (-1)\ln(x) - \frac{1}{2} & \\ & = & \ln\left(x^{-1}\right) - \frac{1}{2} & \text{Power Rule} \\ & = & \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) & \text{Since } \frac{1}{2} = \ln\left(e^{1/2}\right) \\ & = & \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) & \\ & = & \ln\left(\dfrac{x^{-1}}{\sqrt{e}}\right) & \text{Quotient Rule} \\ & = & \ln\left(\dfrac{1}{x\sqrt{e}}\right) & \end{array}$$

□

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, if we are interested in rewriting an expression as a single logarithm, we apply log properties following the usual order of operations: deal with multiples of logs first with the Power Rule, then deal with addition and subtraction using the Product and Quotient Rules, respectively. Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1, \infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty, -1) \cup (1, \infty)$. We will need to keep this in mind when we solve equations involving logarithms in Section 6.4 - it is precisely for this reason we will have to check for extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 6.7. (Change of Base Formulas) Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

The proofs of the Change of Base formulas are a result of the other properties studied in this section. If we start with $b^{x \log_b(a)}$ and use the Power Rule in the exponent to rewrite $x \log_b(a)$ as $\log_b\left(a^x\right)$ and then apply one of the Inverse Properties in Theorem 6.3, we get

$$b^{x \log_b(a)} = b^{\log_b\left(a^x\right)} = a^x,$$

as required. To verify the logarithmic form of the property, we also use the Power Rule and an Inverse Property. We note that

$$\log_a(x) \cdot \log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x),$$

and we get the result by dividing through by $\log_b(a)$. Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$. While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus.[4] What Theorem 6.7 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base - be it 10, e, 42, or 117. Your Calculus teacher will have more to say about this when the time comes.

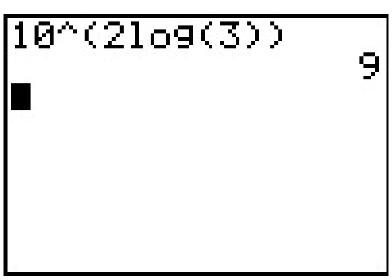
Example 6.2.3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a calculator, as appropriate.

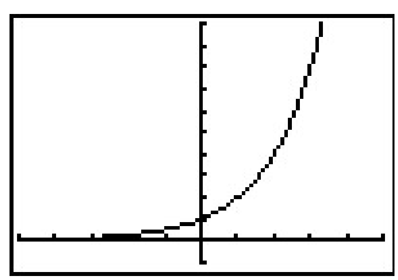
1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution.

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter in the calculator produces an answer of 9 as required.

2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. To verify this on our calculator, we can graph $f(x) = 2^x$ and $g(x) = e^{x\ln(2)}$. Their graphs are indistinguishable which provides evidence that they are the same function.



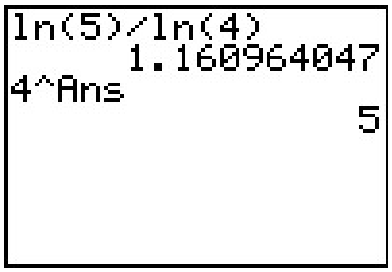


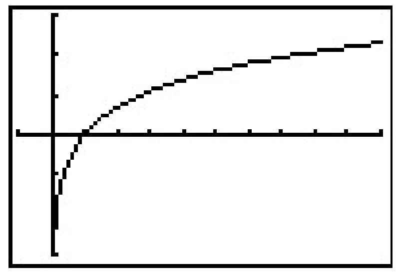
$y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

[4] The authors feel so strongly about showing students that every property of logarithms comes from and corresponds to a property of exponents that we have broken tradition with the vast majority of other authors in this field. This isn't the first time this happened, and it certainly won't be the last.

3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this in the calculator gives $\frac{\ln(5)}{\ln(4)} \approx 1.16$. How do we check this really is the value of $\log_4(5)$? By definition, $\log_4(5)$ is the exponent we put on 4 to get 5. The calculator confirms this.[5]

4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.





$y = f(x) = \ln(x)$ and $y = g(x) = \frac{\log(x)}{\log(e)}$ □

[5] Which means if it is lying to us about the first answer it gave us, at least it is being consistent.

6.2.1 Exercises

In Exercises 1 - 15, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\ln(x^3y^2)$
2. $\log_2\left(\dfrac{128}{x^2+4}\right)$
3. $\log_5\left(\dfrac{z}{25}\right)^3$
4. $\log(1.23 \times 10^{37})$
5. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$
6. $\log_5\left(x^2-25\right)$
7. $\log_{\sqrt{2}}\left(4x^3\right)$
8. $\log_{\frac{1}{3}}(9x(y^3-8))$
9. $\log\left(1000x^3y^5\right)$
10. $\log_3\left(\dfrac{x^2}{81y^4}\right)$
11. $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$
12. $\log_6\left(\dfrac{216}{x^3y}\right)^4$
13. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$
14. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$
15. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 16 - 29, use the properties of logarithms to write the expression as a single logarithm.

16. $4\ln(x)+2\ln(y)$
17. $\log_2(x)+\log_2(y)-\log_2(z)$
18. $\log_3(x)-2\log_3(y)$
19. $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$
20. $2\ln(x)-3\ln(y)-4\ln(z)$
21. $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$
22. $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$
23. $\log_5(x)-3$
24. $3-\log(x)$
25. $\log_7(x)+\log_7(x-3)-2$
26. $\ln(x)+\frac{1}{2}$
27. $\log_2(x)+\log_4(x)$
28. $\log_2(x)+\log_4(x-1)$
29. $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

In Exercises 30 - 33, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

30. 7^{x-1} to base e

31. $\log_3(x+2)$ to base 10

32. $\left(\dfrac{2}{3}\right)^x$ to base e

33. $\log(x^2+1)$ to base e

In Exercises 34 - 39, use the appropriate change of base formula to approximate the logarithm.

34. $\log_3(12)$

35. $\log_5(80)$

36. $\log_6(72)$

37. $\log_4\left(\dfrac{1}{10}\right)$

38. $\log_{\frac{3}{5}}(1000)$

39. $\log_{\frac{2}{3}}(50)$

40. Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.

41. Prove the Quotient Rule and Power Rule for Logarithms.

42. Give numerical examples to show that, in general,

    (a) $\log_b(x+y) \neq \log_b(x) + \log_b(y)$
    (b) $\log_b(x-y) \neq \log_b(x) - \log_b(y)$
    (c) $\log_b\left(\dfrac{x}{y}\right) \neq \dfrac{\log_b(x)}{\log_b(y)}$

43. The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction
$$HA \rightleftharpoons H^+ + A^-.$$
The acid disassociation constant, K_a, is given by
$$K_\alpha = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$
where the square brackets denote the concentrations just as they did in Exercise 77 in Section 6.1. The symbol pK_a is defined similarly to pH in that p$K_a = -\log(K_a)$. Using the definition of pH from Exercise 77 and the properties of logarithms, derive the Henderson-Hasselbalch Equation which states
$$\text{pH} = \text{p}K_a + \log\frac{[A^-]}{[HA]}$$

44. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?

45. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

6.2.2 Answers

1. $3\ln(x)+2\ln(y)$
2. $7-\log_2(x^2+4)$
3. $3\log_5(z)-6$
4. $\log(1.23)+37$
5. $\frac{1}{2}\ln(z)-\ln(x)-\ln(y)$
6. $\log_5(x-5)+\log_5(x+5)$
7. $3\log_{\sqrt{2}}(x)+4$
8. $-2+\log_{\frac{1}{3}}(x)+\log_{\frac{1}{3}}(y-2)+\log_{\frac{1}{3}}(y^2+2y+4)$
9. $3+3\log(x)+5\log(y)$
10. $2\log_3(x)-4-4\log_3(y)$
11. $\frac{1}{4}\ln(x)+\frac{1}{4}\ln(y)-\frac{1}{4}-\frac{1}{4}\ln(z)$
12. $12-12\log_6(x)-4\log_6(y)$
13. $\frac{5}{3}+\log(x)+\frac{1}{2}\log(y)$
14. $-2+\frac{2}{3}\log_{\frac{1}{2}}(x)-\log_{\frac{1}{2}}(y)-\frac{1}{2}\log_{\frac{1}{2}}(z)$
15. $\frac{1}{3}\ln(x)-\ln(10)-\frac{1}{2}\ln(y)-\frac{1}{2}\ln(z)$
16. $\ln(x^4y^2)$
17. $\log_2\left(\frac{xy}{z}\right)$
18. $\log_3\left(\frac{x}{y^2}\right)$
19. $\log_3\left(\frac{\sqrt{x}}{y^2z}\right)$
20. $\ln\left(\frac{x^2}{y^3z^4}\right)$
21. $\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
22. $\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$
23. $\log_5\left(\frac{x}{125}\right)$
24. $\log\left(\frac{1000}{x}\right)$
25. $\log_7\left(\frac{x(x-3)}{49}\right)$
26. $\ln\left(x\sqrt{e}\right)$
27. $\log_2\left(x^{3/2}\right)$
28. $\log_2\left(x\sqrt{x-1}\right)$
29. $\log_2\left(\frac{x}{x-1}\right)$
30. $7^{x-1}=e^{(x-1)\ln(7)}$
31. $\log_3(x+2)=\frac{\log(x+2)}{\log(3)}$
32. $\left(\frac{2}{3}\right)^x=e^{x\ln\left(\frac{2}{3}\right)}$
33. $\log(x^2+1)=\frac{\ln(x^2+1)}{\ln(10)}$
34. $\log_3(12)\approx 2.26186$
35. $\log_5(80)\approx 2.72271$
36. $\log_6(72)\approx 2.38685$
37. $\log_4\left(\frac{1}{10}\right)\approx -1.66096$
38. $\log_{\frac{3}{5}}(1000)\approx -13.52273$
39. $\log_{\frac{2}{3}}(50)\approx -9.64824$

6.3 Exponential Equations and Inequalities

In this section we will develop techniques for solving equations involving exponential functions. Suppose, for instance, we wanted to solve the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 6.4, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution. Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 6.3 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$. This makes sense because, after all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 6.7, to give us something more calculator friendly,[1] say $\log_2(129) = \frac{\ln(129)}{\ln(2)}$. Another way to arrive at this answer is as follows

$$\begin{array}{rcll} 2^x & = & 129 & \\ \ln\left(2^x\right) & = & \ln(129) & \text{Take the natural log of both sides.} \\ x\ln(2) & = & \ln(129) & \text{Power Rule} \\ x & = & \dfrac{\ln(129)}{\ln(2)} & \end{array}$$

'Taking the natural log' of both sides is akin to squaring both sides: since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[2] Also note that we treat $\ln(2)$ as any other non-zero real number and divide it through[3] to isolate the variable x. We summarize below the two common ways to solve exponential equations, motivated by our examples.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

Example 6.3.1. Solve the following equations. Check your answer graphically using a calculator.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

Solution.

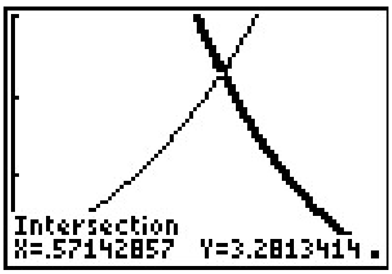
[1] You can use natural logs or common logs. We choose natural logs. (In Calculus, you'll learn these are the most 'mathy' of the logarithms.)

[2] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 6.4.

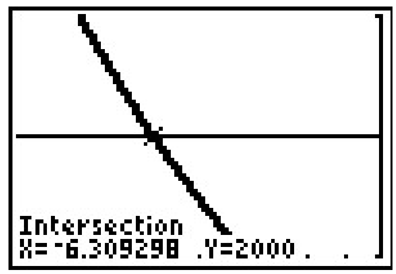
[3] Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$. Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$. To check graphically, we set $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x = \frac{4}{7} \approx 0.5714$.

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$. Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$. Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ to get $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$. On the calculator, we graph $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ and find that they intersect at $x = -\frac{10\ln(2)}{\ln(3)} \approx -6.3093$.



$y = f(x) = 2^{3x}$ and
$\boldsymbol{y = g(x) = 16^{1-x}}$



$y = f(x) = 2000$ and
$\boldsymbol{y = g(x) = 1000 \cdot 3^{-0.1x}}$

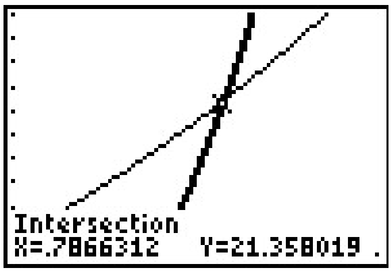
3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$. Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$. The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants. The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{aligned}
(x+2)\ln(3) &= 2x\ln(7) \\
x\ln(3) + 2\ln(3) &= 2x\ln(7) \\
2\ln(3) &= 2x\ln(7) - x\ln(3) \\
2\ln(3) &= x(2\ln(7) - \ln(3)) \quad \text{Factor.} \\
x &= \tfrac{2\ln(3)}{2\ln(7)-\ln(3)}
\end{aligned}$$

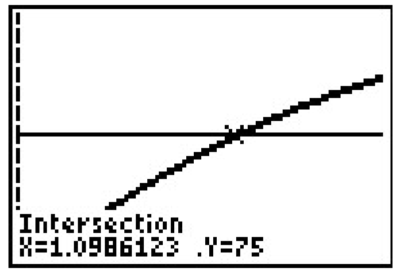
Graphing $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ on the calculator, we see that these two graphs intersect at $x = \frac{2\ln(3)}{2\ln(7)-\ln(3)} \approx 0.7866$.

4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential. To that end, we clear denominators and get $75\left(1 + 3e^{-2t}\right) = 100$. From this we get $75 + 225e^{-2t} = 100$, which leads to $225e^{-2t} = 25$, and finally, $e^{-2t} = \frac{1}{9}$. Taking the natural log of both sides

gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. We can also use the Power Rule to write $\ln\left(\frac{1}{9}\right) = -\ln(9)$. Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?) The calculator confirms the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$ intersect at $x = \ln(3) \approx 1.099$.



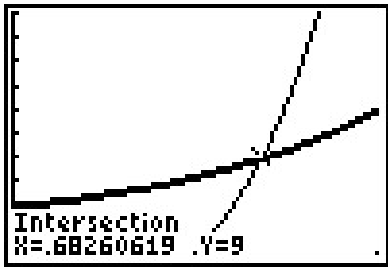
$y = f(x) = 9 \cdot 3^x$ and
$\boldsymbol{y = g(x) = 7^{2x}}$



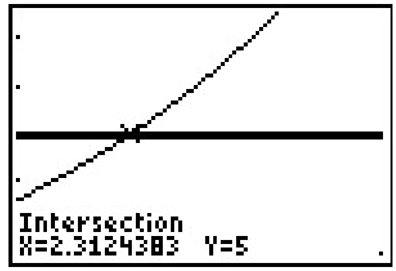
$y = f(x) = 75$ and
$\boldsymbol{y = g(x) = \frac{100}{1+3e^{-2x}}}$

5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$. Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs. If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'. Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$. Since $5^x = -2$ has no real solution, (Why not?) we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln(5^x) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$. On the calculator, we see the graphs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x = \frac{\ln(3)}{\ln(5)} \approx 0.6826$.

6. At first, it's unclear how to proceed with $\frac{e^x - e^{-x}}{2} = 5$, besides clearing the denominator to obtain $e^x - e^{-x} = 10$. Of course, if we rewrite $e^{-x} = \frac{1}{e^x}$, we see we have another denominator lurking in the problem: $e^x - \frac{1}{e^x} = 10$. Clearing this denominator gives us $e^{2x} - 1 = 10e^x$, and once again, we have an equation with three terms where the exponent on one term is exactly twice that of another - a 'quadratic in disguise.' If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$, we obtain by the quadratic formula $u = 5 \pm \sqrt{26}$. From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$, but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$. If we graph $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$, we see that the graphs intersect at $x = \ln\left(5 + \sqrt{26}\right) \approx 2.312$



$y = f(x) = 25^x$ and
$\boldsymbol{y = g(x) = 5^x + 6}$



$y = f(x) = \frac{e^x - e^{-x}}{2}$ and
$\boldsymbol{y = g(x) = 5}$

□

The authors would be remiss not to mention that Example 6.3.1 still holds great educational value. Much can be learned about logarithms and exponentials by verifying the solutions obtained in Example 6.3.1 analytically. For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and obtain

$$\begin{array}{rcll}
2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_3(2)} & \text{Change of Base} \\
2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\
2000 & \stackrel{\checkmark}{=} & 2000 &
\end{array}$$

The other solutions can be verified by using a combination of log and inverse properties. Some fall out quite quickly, while others are more involved. We leave them to the reader.

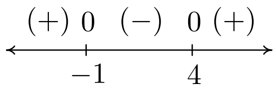
Since exponential functions are continuous on their domains, the Intermediate Value Theorem 3.1 applies. As with the algebraic functions in Section 5.3, this allows us to solve inequalities using sign diagrams as demonstrated below.

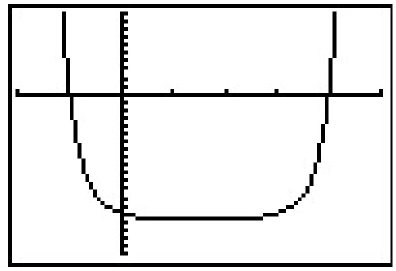
Example 6.3.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $xe^{2x} < 4x$

Solution.

1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$. The domain of r is all real numbers, so in order to construct our sign diagram, we need to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by the one-to-one property of exponential functions, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$, is on or above the x-axis.





$y = r(x) = 2^{x^2-3x} - 16$

2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned}
\frac{e^x}{e^x-4} &\leq 3 \\
\frac{e^x}{e^x-4} - 3 &\leq 0 \\
\frac{e^x}{e^x-4} - \frac{3\left(e^x-4\right)}{e^x-4} &\leq 0 \quad \text{Common denomintors.} \\
\frac{12-2e^x}{e^x-4} &\leq 0
\end{aligned}$$

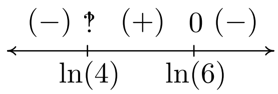
We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[4] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. While the prospect of determining the sign of $r\left(\ln(3)\right)$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

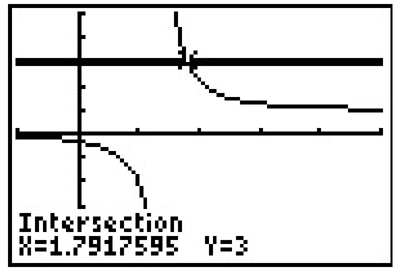
$$r\left(\ln(3)\right) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6$$

We determine the signs of $r\left(\ln(5)\right)$ and $r\left(\ln(7)\right)$ similarly.[5] From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$. Using the calculator, we see the graph of $f(x) = \frac{e^x}{e^x-4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x = \ln(6) \approx 1.792$.

[4]This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

[5]We could, of course, use the calculator, but what fun would that be?



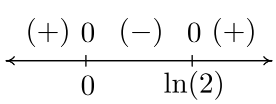


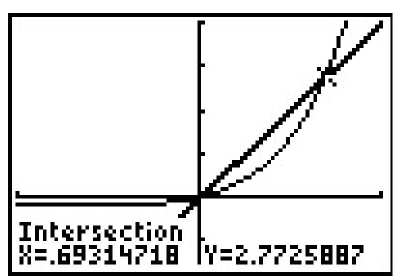
$y = f(x) = \frac{e^x}{e^x - 4}$
$\boldsymbol{y = g(x) = 3}$

3. As before, we start solving $xe^{2x} < 4x$ by getting 0 on one side of the inequality, $xe^{2x} - 4x < 0$. We set $r(x) = xe^{2x} - 4x$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers. Setting $r(x) = 0$ produces $xe^{2x} - 4x = 0$. We factor to get $x\left(e^{2x} - 4\right) = 0$ which gives $x = 0$ or $e^{2x} - 4 = 0$. To solve the latter, we isolate the exponential and take logs to get $2x = \ln(4)$, or $x = \frac{\ln(4)}{2} = \ln(2)$. (Can you explain the last equality using properties of logs?) As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating,[6] we get

$$\begin{array}{rcll} r\left(\ln\left(\frac{1}{2}\right)\right) & = & \ln\left(\frac{1}{2}\right) e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\frac{1}{2}\right) & \\ & = & \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\frac{1}{2}\right) & \text{Power Rule} \\ & = & \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\frac{1}{2}\right) & \\ & = & \frac{1}{4}\ln\left(\frac{1}{2}\right) - 4\ln\left(\frac{1}{2}\right) = -\frac{15}{4}\ln\left(\frac{1}{2}\right) & \end{array}$$

Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$, so $r(x) < 0$ on $(0, \ln(2))$. The calculator confirms that the graph of $f(x) = xe^{2x}$ is below the graph of $g(x) = 4x$ on these intervals.[7]





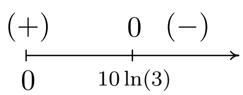
$y = f(x) = xe^{2x}$ and $\boldsymbol{y = g(x) = 4x}$

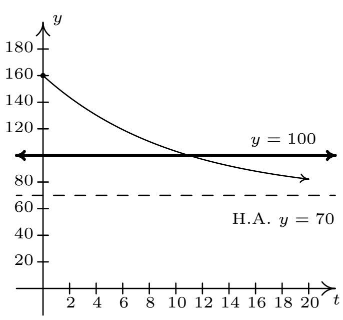
□

[6] A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.
[7] Note: $\ln(2) \approx 0.693$.

Example 6.3.3. Recall from Example 6.1.2 that the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

Solution. We need to find when $T(t) > 100$, or in other words, we need to solve the inequality $70 + 90e^{-0.1t} > 100$. Getting 0 on one side of the inequality, we have $90e^{-0.1t} - 30 > 0$, and we set $r(t) = 90e^{-0.1t} - 30$. The domain of r is artificially restricted due to the context of the problem to $[0, \infty)$, so we proceed to find the zeros of r. Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which, after a quick application of the Power Rule leaves us with $t = 10\ln(3)$. If we wish to avoid using the calculator to choose test values, we note that since $1 < 3$, $0 = \ln(1) < \ln(3)$ so that $10\ln(3) > 0$. So we choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $10\ln(3) < 10\ln(4)$, so the latter is our choice of a test value for the interval $(10\ln(3), \infty)$. Our sign diagram is below, and next to it is our graph of $y = T(t)$ from Example 6.1.2 with the horizontal line $y = 100$.





$y = T(t)$

In order to interpret what this means in the context of the real world, we need a reasonable approximation of the number $10\ln(3) \approx 10.986$. This means it takes approximately 11 minutes for the coffee to cool to 100°F. Until then, the coffee is warmer than that.[8] □

We close this section by finding the inverse of a function which is a composition of a rational function with an exponential function.

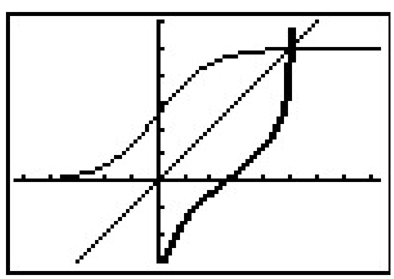
Example 6.3.4. The function $f(x) = \dfrac{5e^x}{e^x + 1}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution. We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

[8]Critics may point out that since we needed to use the calculator to interpret our answer anyway, why not use it earlier to simplify the computations? It is a fair question which we answer unfairly: it's our book.

$$\begin{aligned}
y &= \frac{5e^x}{e^x+1} \\
x &= \frac{5e^y}{e^y+1} && \text{Switch } x \text{ and } y \\
x\left(e^y+1\right) &= 5e^y \\
xe^y + x &= 5e^y \\
x &= 5e^y - xe^y \\
x &= e^y(5-x) \\
e^y &= \frac{x}{5-x} \\
\ln\left(e^y\right) &= \ln\left(\frac{x}{5-x}\right) \\
y &= \ln\left(\frac{x}{5-x}\right)
\end{aligned}$$

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this to the reader. To verify our solution graphically, we graph $y = f(x) = \frac{5e^x}{e^x+1}$ and $y = g(x) = \ln\left(\frac{x}{5-x}\right)$ on the same set of axes and observe the symmetry about the line $y = x$. Note the domain of f is the range of g and vice-versa.



$y = f(x) = \frac{5e^x}{e^x+1}$ and $\boldsymbol{y = g(x) = \ln\left(\frac{x}{5-x}\right)}$

□

6.3.1 EXERCISES

In Exercises 1 - 33, solve the equation analytically.

1. $2^{4x} = 8$
2. $3^{(x-1)} = 27$
3. $5^{2x-1} = 125$
4. $4^{2x} = \frac{1}{2}$
5. $8^x = \frac{1}{128}$
6. $2^{(x^3-x)} = 1$
7. $3^{7x} = 81^{4-2x}$
8. $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
9. $3^{2x} = 5$
10. $5^{-x} = 2$
11. $5^x = -2$
12. $3^{(x-1)} = 29$
13. $(1.005)^{12x} = 3$
14. $e^{-5730k} = \frac{1}{2}$
15. $2000e^{0.1t} = 4000$
16. $500\left(1 - e^{2x}\right) = 250$
17. $70 + 90e^{-0.1t} = 75$
18. $30 - 6e^{-0.1x} = 20$
19. $\dfrac{100e^x}{e^x + 2} = 50$
20. $\dfrac{5000}{1 + 2e^{-3t}} = 2500$
21. $\dfrac{150}{1 + 29e^{-0.8t}} = 75$
22. $25\left(\frac{4}{5}\right)^x = 10$
23. $e^{2x} = 2e^x$
24. $7e^{2x} = 28e^{-6x}$
25. $3^{(x-1)} = 2^x$
26. $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
27. $7^{3+7x} = 3^{4-2x}$
28. $e^{2x} - 3e^x - 10 = 0$
29. $e^{2x} = e^x + 6$
30. $4^x + 2^x = 12$
31. $e^x - 3e^{-x} = 2$
32. $e^x + 15e^{-x} = 8$
33. $3^x + 25 \cdot 3^{-x} = 10$

In Exercises 34 - 39, solve the inequality analytically.

34. $e^x > 53$
35. $1000\,(1.005)^{12t} \geq 3000$
36. $2^{(x^3-x)} < 1$
37. $25\left(\frac{4}{5}\right)^x \geq 10$
38. $\dfrac{150}{1 + 29e^{-0.8t}} \leq 130$
39. $70 + 90e^{-0.1t} \leq 75$

In Exercises 40 - 45, use your calculator to help you solve the equation or inequality.

40. $2^x = x^2$
41. $e^x = \ln(x) + 5$
42. $e^{\sqrt{x}} = x + 1$
43. $e^{-x} - xe^{-x} \geq 0$
44. $3^{(x-1)} < 2^x$
45. $e^x < x^3 - x$
46. Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram. Use this technique to solve the inequalities in Exercises 34 - 39. (NOTE: Isolate the exponential function first!)
47. Compute the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.

48. In Example 6.3.4, we found that the inverse of $f(x) = \dfrac{5e^x}{e^x+1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5-x}\right)$ but we left a few loose ends for you to tie up.

    (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) Let $g(x) = \dfrac{5x}{x+1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. (We know this is true in general by Exercise 31 in Section 5.2, but it's nice to see a specific example of the property.)

49. With the help of your classmates, solve the inequality $e^x > x^n$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = e^x$ grows versus any polynomial?

6.3.2 Answers

1. $x = \frac{3}{4}$
2. $x = 4$
3. $x = 2$
4. $x = -\frac{1}{4}$
5. $x = -\frac{7}{3}$
6. $x = -1,\ 0,\ 1$
7. $x = \frac{16}{15}$
8. $x = -\frac{2}{11}$
9. $x = \frac{\ln(5)}{2\ln(3)}$
10. $x = -\frac{\ln(2)}{\ln(5)}$
11. No solution.
12. $x = \frac{\ln(29)+\ln(3)}{\ln(3)}$
13. $x = \frac{\ln(3)}{12\ln(1.005)}$
14. $k = \frac{\ln\left(\frac{1}{2}\right)}{-5730} = \frac{\ln(2)}{5730}$
15. $t = \frac{\ln(2)}{0.1} = 10\ln(2)$
16. $x = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2)$
17. $t = \frac{\ln\left(\frac{1}{18}\right)}{-0.1} = 10\ln(18)$
18. $x = -10\ln\left(\frac{5}{3}\right) = 10\ln\left(\frac{3}{5}\right)$
19. $x = \ln(2)$
20. $t = \frac{1}{3}\ln(2)$
21. $t = \frac{\ln\left(\frac{1}{29}\right)}{-0.8} = \frac{5}{4}\ln(29)$
22. $x = \frac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)} = \frac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}$
23. $x = \ln(2)$
24. $x = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$
25. $x = \frac{\ln(3)}{\ln(3)-\ln(2)}$
26. $x = \frac{\ln(3)+5\ln\left(\frac{1}{2}\right)}{\ln(3)-\ln\left(\frac{1}{2}\right)} = \frac{\ln(3)-5\ln(2)}{\ln(3)+\ln(2)}$
27. $x = \frac{4\ln(3)-3\ln(7)}{7\ln(7)+2\ln(3)}$
28. $x = \ln(5)$
29. $x = \ln(3)$
30. $x = \frac{\ln(3)}{\ln(2)}$
31. $x = \ln(3)$
32. $x = \ln(3),\ \ln(5)$
33. $x = \frac{\ln(5)}{\ln(3)}$
34. $(\ln(53), \infty)$
35. $\left[\frac{\ln(3)}{12\ln(1.005)}, \infty\right)$
36. $(-\infty, -1) \cup (0, 1)$
37. $\left(-\infty, \frac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right] = \left(-\infty, \frac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}\right]$
38. $\left(-\infty, \frac{\ln\left(\frac{2}{377}\right)}{-0.8}\right] = \left(-\infty, \frac{5}{4}\ln\left(\frac{377}{2}\right)\right]$
39. $\left[\frac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$
40. $x \approx -0.76666,\ x = 2,\ x = 4$
41. $x \approx 0.01866,\ x \approx 1.7115$
42. $x = 0$
43. $(-\infty, 1]$
44. $\approx (-\infty, 2.7095)$
45. $\approx (2.3217, 4.3717)$
46. $x > \frac{1}{3}(\ln(6)+1)$
47. $f^{-1} = \ln\left(x + \sqrt{x^2+1}\right)$. Both f and f^{-1} have domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

6.4 Logarithmic Equations and Inequalities

In Section 6.3 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from. For example, suppose we wish to solve $\log_2(x) = \log_2(5)$. Theorem 6.4 tells us that the *only* solution to this equation is $x = 5$. Now suppose we wish to solve $\log_2(x) = 3$. If we want to use Theorem 6.4, we need to rewrite 3 as a logarithm base 2. We can use Theorem 6.3 to do just that: $3 = \log_2\left(2^3\right) = \log_2(8)$. Our equation then becomes $\log_2(x) = \log_2(8)$ so that $x = 8$. However, we could have arrived at the same answer, in fewer steps, by using Theorem 6.3 to rewrite the equation $\log_2(x) = 3$ as $2^3 = x$, or $x = 8$. We summarize the two common ways to solve log equations below.

Steps for Solving an Equation involving Logarithmic Functions

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate the arguments of the log functions.
   (b) Otherwise, rewrite the log equation as an exponential equation.

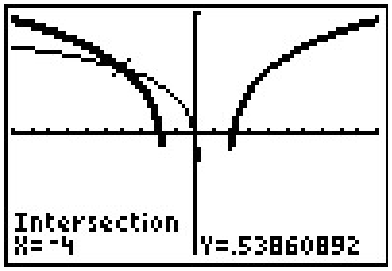
Example 6.4.1. Solve the following equations. Check your solutions graphically using a calculator.

1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(x-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2x) = 1 - \log_7(3-x)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(x+1) = 2\log_2(x)$

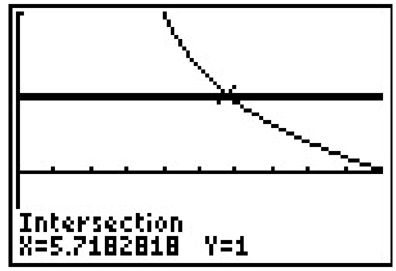
Solution.

1. Since we have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, we equate what's inside the logs to get $1 - 3x = x^2 - 3$. Solving $x^2 + 3x - 4 = 0$ gives $x = -4$ and $x = 1$. To check these answers using the calculator, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$ and we see they intersect only at $x = -4$. To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x = 1$ is not in the domain of the original equation, and is not a solution.
2. Our first objective in solving $2 - \ln(x-3) = 1$ is to isolate the logarithm. We get $\ln(x-3) = 1$, which, as an exponential equation, is $e^1 = x - 3$. We get our solution $x = e + 3$. On the calculator, we see the graph of $f(x) = 2 - \ln(x-3)$ intersects the graph of $g(x) = 1$ at $x = e + 3 \approx 5.718$.

[1] They do, however, represent the same **family** of complex numbers. We stop ourselves at this point and refer the reader to a good course in Complex Variables.

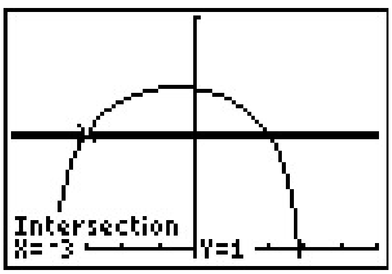


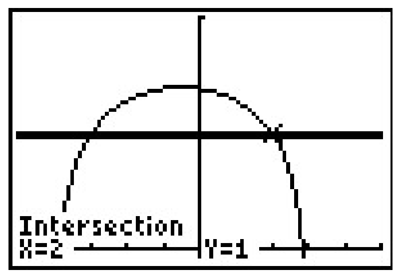
$y = f(x) = \log_{117}(1-3x)$ and
$\boldsymbol{y = g(x) = \log_{117}\left(x^2-3\right)}$



$y = f(x) = 2 - \ln(x-3)$ and
$\boldsymbol{y = g(x) = 1}$

3. We can start solving $\log_6(x+4) + \log_6(3-x) = 1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6\left[(x+4)(3-x)\right] = 1$. Rewriting this as an exponential equation, we get $6^1 = (x+4)(3-x)$. This reduces to $x^2+x-6=0$, which gives $x=-3$ and $x=2$. Graphing $y = f(x) = \frac{\ln(x+4)}{\ln(6)} + \frac{\ln(3-x)}{\ln(6)}$ and $y = g(x) = 1$, we see they intersect twice, at $x=-3$ and $x=2$.





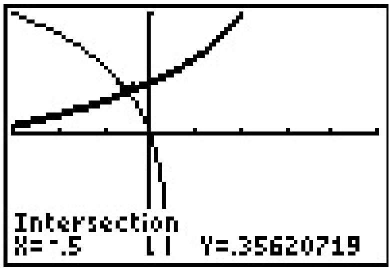
$y = f(x) = \log_6(x+4) + \log_6(3-x)$ and $\boldsymbol{y = g(x) = 1}$

4. Taking a cue from the previous problem, we begin solving $\log_7(1-2x) = 1 - \log_7(3-x)$ by first collecting the logarithms on the same side, $\log_7(1-2x) + \log_7(3-x) = 1$, and then using the Product Rule to get $\log_7[(1-2x)(3-x)] = 1$. Rewriting this as an exponential equation gives $7^1 = (1-2x)(3-x)$ which gives the quadratic equation $2x^2-7x-4=0$. Solving, we find $x = -\frac{1}{2}$ and $x=4$. Graphing, we find $y = f(x) = \frac{\ln(1-2x)}{\ln(7)}$ and $y = g(x) = 1 - \frac{\ln(3-x)}{\ln(7)}$ intersect only at $x = -\frac{1}{2}$. Checking $x=4$ in the original equation produces $\log_7(-7) = 1 - \log_7(-1)$, which is a clear domain violation.

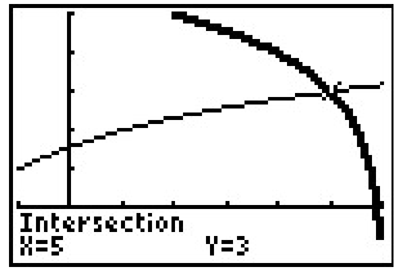
5. Starting with $\log_2(x+3) = \log_2(6-x) + 3$, we gather the logarithms to one side and get $\log_2(x+3) - \log_2(6-x) = 3$. We then use the Quotient Rule and convert to an exponential equation

$$\log_2\left(\frac{x+3}{6-x}\right) = 3 \iff 2^3 = \frac{x+3}{6-x}$$

This reduces to the linear equation $8(6-x) = x+3$, which gives us $x=5$. When we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$, we find they intersect at $x=5$.



$y = f(x) = \log_7(1-2x)$ and
$\boldsymbol{y = g(x) = 1 - \log_7(3-x)}$



$y = f(x) = \log_2(x+3)$ and
$\boldsymbol{y = g(x) = \log_2(6-x) + 3}$

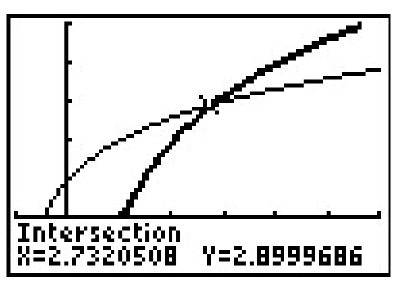
6. Starting with $1 + 2\log_4(x+1) = 2\log_2(x)$, we gather the logs to one side to get the equation $1 = 2\log_2(x) - 2\log_4(x+1)$. Before we can combine the logarithms, however, we need a common base. Since 4 is a power of 2, we use change of base to convert

$$\log_4(x+1) = \frac{\log_2(x+1)}{\log_2(4)} = \frac{1}{2}\log_2(x+1)$$

Hence, our original equation becomes

$$\begin{array}{rcll} 1 & = & 2\log_2(x) - 2\left(\frac{1}{2}\log_2(x+1)\right) & \\ 1 & = & 2\log_2(x) - \log_2(x+1) & \\ 1 & = & \log_2\left(x^2\right) - \log_2(x+1) & \text{Power Rule} \\ 1 & = & \log_2\left(\dfrac{x^2}{x+1}\right) & \text{Quotient Rule} \end{array}$$

Rewriting this in exponential form, we get $\frac{x^2}{x+1} = 2$ or $x^2 - 2x - 2 = 0$. Using the quadratic formula, we get $x = 1 \pm \sqrt{3}$. Graphing $f(x) = 1 + \frac{2\ln(x+1)}{\ln(4)}$ and $g(x) = \frac{2\ln(x)}{\ln(2)}$, we see the graphs intersect only at $x = 1 + \sqrt{3} \approx 2.732$. The solution $x = 1 - \sqrt{3} < 0$, which means if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined.



$y = f(x) = 1 + 2\log_4(x+1)$ and $\boldsymbol{y = g(x) = 2\log_2(x)}$

□

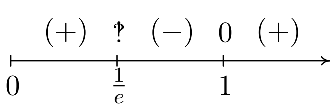
If nothing else, Example 6.4.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically, extraneous solutions are easy to spot - any supposed solution which causes a negative number inside a logarithm needs to be discarded. As with the equations in Example 6.3.1, much can be learned from checking all of the answers in Example 6.4.1 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

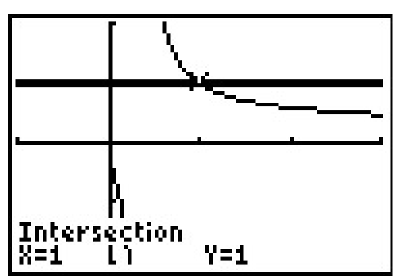
Example 6.4.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $\dfrac{1}{\ln(x)+1} \leq 1$
2. $(\log_2(x))^2 < 2\log_2(x) + 3$
3. $x\log(x+1) \geq x$

Solution.

1. We start solving $\frac{1}{\ln(x)+1} \leq 1$ by getting 0 on one side of the inequality: $\frac{1}{\ln(x)+1} - 1 \leq 0$. Getting a common denominator yields $\frac{1}{\ln(x)+1} - \frac{\ln(x)+1}{\ln(x)+1} \leq 0$ which reduces to $\frac{-\ln(x)}{\ln(x)+1} \leq 0$, or $\frac{\ln(x)}{\ln(x)+1} \geq 0$. We define $r(x) = \frac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of r. Due to the appearance of the term $\ln(x)$, we require $x > 0$. In order to keep the denominator away from zero, we solve $\ln(x) + 1 = 0$ so $\ln(x) = -1$, so $x = e^{-1} = \frac{1}{e}$. Hence, the domain of r is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$. To find the zeros of r, we set $r(x) = \frac{\ln(x)}{\ln(x)+1} = 0$ so that $\ln(x) = 0$, and we find $x = e^0 = 1$. In order to determine test values for r without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of e. Since $e \approx 2.718 > 1$, $0 < \frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}} < 1 < e$. To determine the sign of $r\left(\frac{1}{e^2}\right)$, we use the fact that $\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$, and find $r\left(\frac{1}{e^2}\right) = \frac{-2}{-2+1} = 2$, which is $(+)$. The rest of the test values are determined similarly. From our sign diagram, we find the solution to be $\left(0, \frac{1}{e}\right) \cup [1, \infty)$. Graphing $f(x) = \frac{1}{\ln(x)+1}$ and $g(x) = 1$, we see the graph of f is below the graph of g on the solution intervals, and that the graphs intersect at $x = 1$.





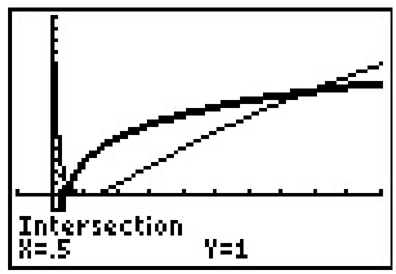
$y = f(x) = \frac{1}{\ln(x)+1}$ and $\boldsymbol{y = g(x) = 1}$

[2]Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

2. Moving all of the nonzero terms of $(\log_2(x))^2 < 2\log_2(x) + 3$ to one side of the inequality, we have $(\log_2(x))^2 - 2\log_2(x) - 3 < 0$. Defining $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3$, we get the domain of r is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of r, we set $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3 = 0$ which results in a 'quadratic in disguise.' We set $u = \log_2(x)$ so our equation becomes $u^2 - 2u - 3 = 0$ which gives us $u = -1$ and $u = 3$. Since $u = \log_2(x)$, we get $\log_2(x) = -1$, which gives us $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which yields $x = 2^3 = 8$. We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$, and from our sign diagram, we see $r(x) < 0$ on $\left(\frac{1}{2}, 8\right)$. Geometrically, we see the graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on the solution interval.

$(+)\ 0\ (-)\ 0\ (+)$

$0 \qquad \frac{1}{2} \qquad 8$

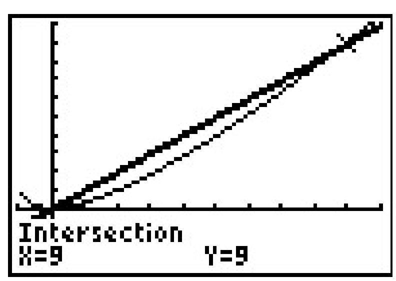


$y = f(x) = (\log_2(x))^2$ and $\boldsymbol{y = g(x) = 2\log_2(x) + 3}$

3. We begin to solve $x\log(x+1) \geq x$ by subtracting x from both sides to get $x\log(x+1) - x \geq 0$. We define $r(x) = x\log(x+1) - x$ and due to the presence of the logarithm, we require $x+1 > 0$, or $x > -1$. To find the zeros of r, we set $r(x) = x\log(x+1) - x = 0$. Factoring, we get $x(\log(x+1) - 1) = 0$, which gives $x = 0$ or $\log(x+1) - 1 = 0$. The latter gives $\log(x+1) = 1$, or $x + 1 = 10^1$, which admits $x = 9$. We select test values x so that $x + 1$ is a power of 10, and we obtain $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$. Our sign diagram gives the solution to be $(-1, 0] \cup [9, \infty)$. The calculator indicates the graph of $y = f(x) = x\log(x+1)$ is above $y = g(x) = x$ on the solution intervals, and the graphs intersect at $x = 0$ and $x = 9$.

$(+)\ 0\ (-)\ 0\ (+)$

$-1 \qquad 0 \qquad 9$



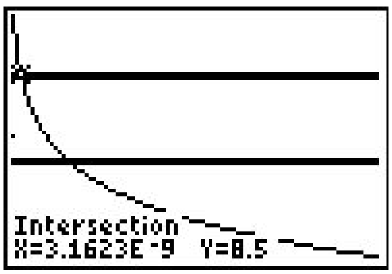
$y = f(x) = x\log(x+1)$ and $\boldsymbol{y = g(x) = x}$

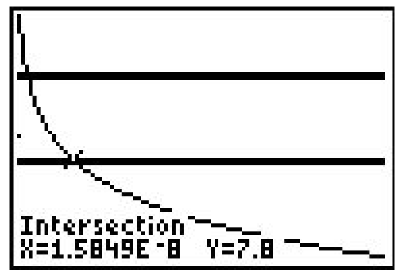
$\square$

Our next example revisits the concept of pH first seen in Exercise 77 in Section 6.1.

Example 6.4.3. In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration, and check your answer using a calculator.

Solution. Recall from Exercise 77 in Section 6.1 that $\mathrm{pH} = -\log[\mathrm{H}^+]$ where $[\mathrm{H}^+]$ is the hydrogen ion concentration in moles per liter. We require $7.8 \leq -\log[\mathrm{H}^+] \leq 8.5$ or $-7.8 \geq \log[\mathrm{H}^+] \geq -8.5$. To solve this compound inequality we solve $-7.8 \geq \log[\mathrm{H}^+]$ and $\log[\mathrm{H}^+] \geq -8.5$ and take the intersection of the solution sets.[3] The former inequality yields $0 < [\mathrm{H}^+] \leq 10^{-7.8}$ and the latter yields $[\mathrm{H}^+] \geq 10^{-8.5}$. Taking the intersection gives us our final answer $10^{-8.5} \leq [\mathrm{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\mathrm{H}^+] \leq 1.58 \times 10^{-8}$.) After carefully adjusting the viewing window on the graphing calculator we see that the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.16 \times 10^{-9}, 1.58 \times 10^{-8}]$.





The graphs of $y = f(x) = -\log(x)$, $\boldsymbol{y = 7.8}$ and $\boldsymbol{y = 8.5}$

□

We close this section by finding an inverse of a one-to-one function which involves logarithms.

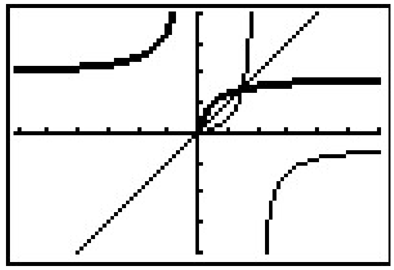
Example 6.4.4. The function $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution. We first write $y = f(x)$ then interchange the x and y and solve for y.

$$\begin{array}{rcll}
y & = & f(x) & \\
y & = & \dfrac{\log(x)}{1-\log(x)} & \\
x & = & \dfrac{\log(y)}{1-\log(y)} & \text{Interchange } x \text{ and } y. \\
x\left(1-\log(y)\right) & = & \log(y) & \\
x - x\log(y) & = & \log(y) & \\
x & = & x\log(y) + \log(y) & \\
x & = & (x+1)\log(y) & \\
\dfrac{x}{x+1} & = & \log(y) & \\
y & = & 10^{\frac{x}{x+1}} & \text{Rewrite as an exponential equation.}
\end{array}$$

[3] Refer to page 4 for a discussion of what this means.

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} on the same viewing window yields



$y = f(x) = \dfrac{\log(x)}{1 - \log(x)}$ and $\boldsymbol{y = g(x) = 10^{\frac{x}{x+1}}}$

$\square$

6.4.1 EXERCISES

In Exercises 1 - 24, solve the equation analytically.

1. $\log(3x-1) = \log(4-x)$
2. $\log_2\left(x^3\right) = \log_2(x)$
3. $\ln\left(8-x^2\right) = \ln(2-x)$
4. $\log_5\left(18-x^2\right) = \log_5(6-x)$
5. $\log_3(7-2x) = 2$
6. $\log_{\frac{1}{2}}(2x-1) = -3$
7. $\ln\left(x^2-99\right) = 0$
8. $\log(x^2-3x) = 1$
9. $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$
10. $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$
11. $-\log(x) = 5.4$
12. $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$
13. $6 - 3\log_5(2x) = 0$
14. $3\ln(x) - 2 = 1 - \ln(x)$
15. $\log_3(x-4) + \log_3(x+4) = 2$
16. $\log_5(2x+1) + \log_5(x+2) = 1$
17. $\log_{169}(3x+7) - \log_{169}(5x-9) = \dfrac{1}{2}$
18. $\ln(x+1) - \ln(x) = 3$
19. $2\log_7(x) = \log_7(2) + \log_7(x+12)$
20. $\log(x) - \log(2) = \log(x+8) - \log(x+2)$
21. $\log_3(x) = \log_{\frac{1}{3}}(x) + 8$
22. $\ln(\ln(x)) = 3$
23. $(\log(x))^2 = 2\log(x) + 15$
24. $\ln(x^2) = (\ln(x))^2$

In Exercises 25 - 30, solve the inequality analytically.

25. $\dfrac{1-\ln(x)}{x^2} < 0$
26. $x\ln(x) - x > 0$
27. $10\log\left(\dfrac{x}{10^{-12}}\right) \geq 90$
28. $5.6 \leq \log\left(\dfrac{x}{10^{-3}}\right) \leq 7.1$
29. $2.3 < -\log(x) < 5.4$
30. $\ln(x^2) \leq (\ln(x))^2$

In Exercises 31 - 34, use your calculator to help you solve the equation or inequality.

31. $\ln(x) = e^{-x}$
32. $\ln(x) = \sqrt[4]{x}$
33. $\ln(x^2+1) \geq 5$
34. $\ln(-2x^3 - x^2 + 13x - 6) < 0$

35. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x+1) < 3$ without a sign diagram. Use this technique to solve the inequalities in Exercises 27 - 29. (Compare this to Exercise 46 in Section 6.3.)

36. Solve $\ln(3-y) - \ln(y) = 2x + \ln(5)$ for y.

37. In Example 6.4.4 we found the inverse of $f(x) = \dfrac{\log(x)}{1-\log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

    (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

    (b) Find the range of f by finding the domain of f^{-1}.

    (c) Let $g(x) = \dfrac{x}{1-x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. (We know this is true in general by Exercise 31 in Section 5.2, but it's nice to see a specific example of the property.)

38. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.

39. Explain the equation in Exercise 10 and the inequality in Exercise 28 above in terms of the Richter scale for earthquake magnitude. (See Exercise 75 in Section 6.1.)

40. Explain the equation in Exercise 12 and the inequality in Exercise 27 above in terms of sound intensity level as measured in decibels. (See Exercise 76 in Section 6.1.)

41. Explain the equation in Exercise 11 and the inequality in Exercise 29 above in terms of the pH of a solution. (See Exercise 77 in Section 6.1.)

42. With the help of your classmates, solve the inequality $\sqrt[n]{x} > \ln(x)$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = \ln(x)$ grows versus any principal n^{th} root function?

6.4.2 Answers

1. $x = \frac{5}{4}$
2. $x = 1$
3. $x = -2$
4. $x = -3,\ 4$
5. $x = -1$
6. $x = \frac{9}{2}$
7. $x = \pm 10$
8. $x = -2,\ 5$
9. $x = -\frac{17}{7}$
10. $x = 10^{1.7}$
11. $x = 10^{-5.4}$
12. $x = 10^{3}$
13. $x = \frac{25}{2}$
14. $x = e^{3/4}$
15. $x = 5$
16. $x = \frac{1}{2}$
17. $x = 2$
18. $x = \frac{1}{e^3 - 1}$
19. $x = 6$
20. $x = 4$
21. $x = 81$
22. $x = e^{e^3}$
23. $x = 10^{-3},\ 10^{5}$
24. $x = 1,\ x = e^2$
25. (e, ∞)
26. (e, ∞)
27. $\left[10^{-3}, \infty\right)$
28. $\left[10^{2.6}, 10^{4.1}\right]$
29. $\left(10^{-5.4}, 10^{-2.3}\right)$
30. $(0, 1] \cup [e^2, \infty)$
31. $x \approx 1.3098$
32. $x \approx 4.177,\ x \approx 5503.665$
33. $\approx (-\infty, -12.1414) \cup (12.1414, \infty)$
34. $\approx (-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$
35. $-\dfrac{1}{2} < x < \dfrac{e^3 - 1}{2}$
36. $y = \dfrac{3}{5e^{2x} + 1}$

38. $f^{-1}(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$. (To see why we rewrite this in this form, see Exercise 51 in Section 11.10.) The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.

6.5 Applications of Exponential and Logarithmic Functions

As we mentioned in Section 6.1, exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, the calculator is often used to express our answers as decimal approximations.

6.5.1 Applications of Exponential Functions

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have \$100 to invest at your local bank and they are offering a whopping 5 % annual percentage interest rate. This means that after one year, the bank will pay *you* 5% of that \$100, or $\$100(0.05) = \$5$ in interest, so you now have \$105.[1] This is in accordance with the formula for *simple interest* which you have undoubtedly run across at some point before.

Equation 6.1. Simple Interest The amount of interest I accrued at an annual rate r on an investment[a] P after t years is

$$I = Prt$$

The amount A in the account after t years is given by

$$A = P + I = P + Prt = P(1 + rt)$$

[a]Called the **principal**

Suppose, however, that six months into the year, you hear of a better deal at a rival bank.[2] Naturally, you withdraw your money and try to invest it at the higher rate there. Since six months is one half of a year, that initial \$100 yields $\$100(0.05)\left(\frac{1}{2}\right) = \$2.50$ in interest. You take your \$102.50 off to the competitor and find out that those restrictions which *may* apply actually do apply to you, and you return to your bank which happily accepts your \$102.50 for the remaining six months of the year. To your surprise and delight, at the end of the year your statement reads \$105.06, not \$105 as you had expected.[3] Where did those extra six cents come from? For the first six months of the year, interest was earned on the original principal of \$100, but for the second six months, interest was earned on \$102.50, that is, you earned interest on your interest. This is the basic concept behind **compound interest**. In the previous discussion, we would say that the interest was compounded twice, or semiannually.[4] If more money can be earned by earning interest on interest already earned, a natural question to ask is what happens if the interest is compounded more often, say 4 times a year, which is every three months, or 'quarterly.' In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1 + rt) = \$100\left(1 + 0.05 \cdot \frac{1}{4}\right) = \$101.25$. We now invest the \$101.25 for the next three

[1]How generous of them!
[2]Some restrictions may apply.
[3]Actually, the final balance should be \$105.0625.
[4]Using this convention, simple interest after one year is the same as compounding the interest only once.

months and find that at the end of the second quarter, we have $A = \$101.25\left(1 + 0.05 \cdot \frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is \$103.79, and, at last, we obtain \$105.08. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound. In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal P at an annual rate r and compound the interest n times per year. This means the money sits in the account $\frac{1}{n}^{\text{th}}$ of a year between compoundings. Let A_k denote the amount in the account after the k^{th} compounding. Then $A_1 = P\left(1 + r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1 + \frac{r}{n}\right)$. After the second compounding, we use A_1 as our new principal and get $A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$. Since we compound the interest n times per year, after t years, we have nt compoundings. We have just derived the general formula for compound interest below.

Equation 6.2. Compounded Interest: If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount A in the account after t years is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 6.2 becomes $A(t) = 100\left(1 + \frac{0.05}{4}\right)^{4t}$ which reduces to $A(t) = 100(1.0125)^{4t}$. To check this new formula against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

Example 6.5.1. Suppose \$2000 is invested in an account which offers 7.125% compounded monthly.

1. Express the amount A in the account as a function of the term of the investment t in years.

2. How much is in the account after 5 years?

3. How long will it take for the initial investment to double?

4. Find and interpret the average rate of change[5] of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution.

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (since interest is compounded *monthly*) into Equation 6.2 yields $A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$.

2. Since t represents the length of the investment in years, we substitute $t = 5$ into $A(t)$ to find $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$. After 5 years, we have approximately \$2852.92.

[5]See Definition 2.3 in Section 2.1.

3. Our initial investment is \$2000, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$. Taking natural logs as in Section 6.3, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.

4. To find the average rate of change of A from the end of the fourth year to the end of the fifth year, we compute $\frac{A(5)-A(4)}{5-4} \approx 195.63$. Similarly, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$. This means that the value of the investment is increasing at a rate of approximately \$195.63 per year between the end of the fourth and fifth years, while that rate jumps to \$1648.21 per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.[6] □

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit.[7] Consider an investment of \$1 invested at 100% interest for 1 year compounded n times a year. Equation 6.2 tells us that the amount of money in the account after 1 year is $A = \left(1 + \frac{1}{n}\right)^n$. Below is a table of values relating n and A.

n	A
1	2
2	2.25
4	≈ 2.4414
12	≈ 2.6130
360	≈ 2.7145
1000	≈ 2.7169
10000	≈ 2.7181
100000	≈ 2.7182

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing. We are witnessing a mathematical 'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so there is less time to build up additional interest. With Calculus, we can show[8] that as $n \to \infty$, $A = \left(1 + \frac{1}{n}\right)^n \to e$, where e is the natural base first presented in Section 6.1. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

Theorem 6.8. If you invest \$1 at 100% interest compounded continuously, then you will have \$$e$ at the end of one year.

[6] In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

[7] Once you've had a semester of Calculus, you'll be able to fully appreciate this very lame pun.

[8] Or define, depending on your point of view.

Using this definition of e and a little Calculus, we can take Equation 6.2 and produce a formula for continuously compounded interest.

Equation 6.3. Continuously Compounded Interest: If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount A in the account after t years is

$$A(t) = Pe^{rt}$$

If we take the scenario of Example 6.5.1 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$ which is about \$24,035.28, whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$ which is about \$24,213.18 - a difference of less than 1%.

Equations 6.2 and 6.3 both use exponential functions to describe the growth of an investment. Curiously enough, the same principles which govern compound interest are also used to model short term growth of populations. In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time.[9] In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Its solution is stated below.

Equation 6.4. Uninhibited Growth: If a population increases according to The Law of Uninhibited Growth, the number of organisms N at time t is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)$$

It is worth taking some time to compare Equations 6.3 and 6.4. In Equation 6.3, we use P to denote the initial investment; in Equation 6.4, we use N_0 to denote the initial population. In Equation 6.3, r denotes the annual interest rate, and so it shouldn't be too surprising that the k in Equation 6.4 corresponds to a growth rate as well. While Equations 6.3 and 6.4 look entirely different, they both represent the same mathematical concept.

Example 6.5.2. In order to perform arthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, N, in thousands, after t days.

Solution. We begin with $N(t) = N_0 e^{kt}$. Since N is to give the number of cells *in thousands*, we have $N_0 = 12$, so $N(t) = 12e^{kt}$. In order to complete the formula, we need to determine the

[9] The average rate of change of a function over an interval was first introduced in Section 2.1. *Instantaneous* rates of change are the business of Calculus, as is mentioned on Page 161.

growth rate k. We know that after one week, the number of cells has grown to five million. Since t measures days and the units of N are in thousands, this translates mathematically to $N(7) = 5000$. We get the equation $12e^{7k} = 5000$ which gives $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$. Hence, $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$. Of course, in practice, we would approximate k to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells. □

Whereas Equations 6.3 and 6.4 model the growth of quantities, we can use equations like them to describe the decline of quantities. One example we've seen already is Example 6.1.1 in Section 6.1. There, the value of a car declined from its purchase price of \$25,000 to nothing at all. Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes. The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays. This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 6.4 with the exception that the rate constant k is negative.

Equation 6.5. Radioactive Decay The amount of a radioactive element A at time t is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } A(t) \text{ at time } t) = k\,A(t)$$

Example 6.5.3. Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 6.5, and that the half-life[10] of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, A, in grams, t days later.

Solution. Since we start with 5 grams initially, Equation 6.5 gives $A(t) = 5e^{kt}$. Since the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Hence, $A(8) = 2.5$ which means $5e^{8k} = 2.5$. Solving, we get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily. Hence, $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$. □

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 6.1.2 of Section 6.1. In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70 + 90e^{-0.1t}$, where t was measured in minutes. In this situation, we know the physical limit of the temperature of the coffee is room temperature,[11] and the differential equation

[10] The time it takes for half of the substance to decay.

[11] The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

which gives rise to our formula for $T(t)$ takes this into account. Whereas the radioactive decay model had a rate of decay at time t directly proportional to the amount of the element which remained at time t, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time t is directly proportional to how much of a temperature gap exists between the coffee at time t and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly. Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and since the physics behind warming and cooling is the same, we combine both cases in the equation below.

Equation 6.6. Newton's Law of Cooling (Warming): The temperature T of an object at time t is given by the formula

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature[a] and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } T(t) \text{ at time } t) = k\, (T(t) - T_a)$$

[a]That is, the temperature of the surroundings.

If we re-examine the situation in Example 6.1.2 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 6.6, $T(t) = 70 + (160 - 70)e^{-0.1t}$ which reduces to the original formula given. The rate constant $k = 0.1$ indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings. Note in Equation 6.6 that the constant k is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if T_0 (the initial temperature of the object) is greater than T_a (the ambient temperature) or vice-versa, as we see in our next example.

Example 6.5.4. A 40°F roast is cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, find a formula for the temperature of the roast T as a function of its time in the oven, t, in hours.

2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

Solution.

1. The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming tells us $T(t) = 350 + (40 - 350)e^{-kt}$, or $T(t) = 350 - 310e^{-kt}$. To determine k, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature function is

$$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$

2. To determine when the roast is done, we set $T(t) = 165$. This gives $350 - 310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. It takes roughly 3 hours and 15 minutes to cook the roast completely. □

If we had taken the time to graph $y = T(t)$ in Example 6.5.4, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion by applying a bit of 'number sense'. As $t \to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of t, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350 -$ very small $(+)$, which indicates the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F.[12] The function T is sometimes called a **limited** growth model, since the function T remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate. The **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

Equation 6.7. Logistic Growth: If a population behaves according to the assumptions of logistic growth, the number of organisms N at time t is given by the equation

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, L is the limiting population,[a] C is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)\,(L - N(t))$$

[a]That is, as $t \to \infty$, $N(t) \to L$

The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumors.[13]

Example 6.5.5. The number of people N, in hundreds, at a local community college who have heard the rumor 'Carl is afraid of Virginia Woolf' can be modeled using the logistic equation

$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

[12]at which point it would be more toast than roast.

[13]Which can be just as damaging as diseases.

where $t \geq 0$ is the number of days after April 1, 2009.

1. Find and interpret $N(0)$.
2. Find and interpret the end behavior of $N(t)$.
3. How long until 4200 people have heard the rumor?
4. Check your answers to 2 and 3 using your calculator.

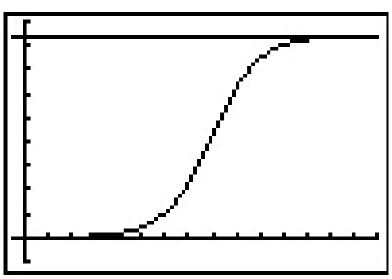
Solution.

1. We find $N(0) = \frac{84}{1+2799e^{0}} = \frac{84}{2800} = \frac{3}{100}$. Since $N(t)$ measures the number of people who have heard the rumor in hundreds, $N(0)$ corresponds to 3 people. Since $t = 0$ corresponds to April 1, 2009, we may conclude that on that day, 3 people have heard the rumor.[14]

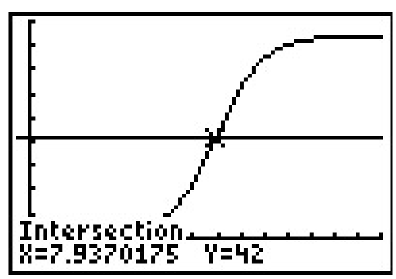
2. We could simply note that $N(t)$ is written in the form of Equation 6.7, and identify $L = 84$. However, to see why the answer is 84, we proceed analytically. Since the domain of N is restricted to $t \geq 0$, the only end behavior of significance is $t \to \infty$. As we've seen before,[15] as $t \to \infty$, we have $1997e^{-t} \to 0^{+}$ and so $N(t) \approx \frac{84}{1+\text{very small }(+)} \approx 84$. Hence, as $t \to \infty$, $N(t) \to 84$. This means that as time goes by, the number of people who will have heard the rumor approaches 8400.

3. To find how long it takes until 4200 people have heard the rumor, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$. It takes around 8 days until 4200 people have heard the rumor.

4. We graph $y = N(x)$ using the calculator and see that the line $y = 84$ is the horizontal asymptote of the graph, confirming our answer to part 2, and the graph intersects the line $y = 42$ at $x = \ln(2799) \approx 7.937$, which confirms our answer to part 3.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 84$



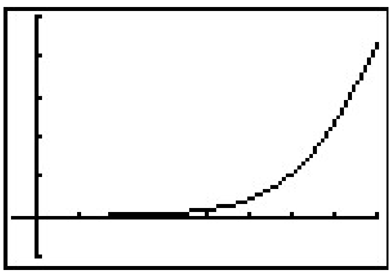
$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 42$

[14] Or, more likely, three people started the rumor. I'd wager Jeff, Jamie, and Jason started it. So much for telling your best friends something in confidence!

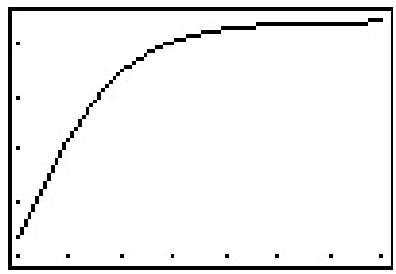
[15] See, for example, Example 6.1.2.

□

If we take the time to analyze the graph of $y = N(x)$ above, we can see graphically how logistic growth combines features of uninhibited and limited growth. The curve seems to rise steeply, then at some point, begins to level off. The point at which this happens is called an **inflection point** or is sometimes called the 'point of diminishing returns'. At this point, even though the function is still increasing, the rate at which it does so begins to decline. It turns out the point of diminishing returns always occurs at half the limiting population. (In our case, when $y = 42$.) While these concepts are more precisely quantified using Calculus, below are two views of the graph of $y = N(x)$, one on the interval $[0, 8]$, the other on $[8, 15]$. The former looks strikingly like uninhibited growth; the latter like limited growth.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $0 \leq x \leq 8$



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $8 \leq x \leq 16$

6.5.2 Applications of Logarithms

Just as many physical phenomena can be modeled by exponential functions, the same is true of logarithmic functions. In Exercises 75, 76 and 77 of Section 6.1, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 6.5.6. The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive[16] password is comprised of letters and numbers only, find the associated information entropy.
2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

Solution.

1. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

[16] That is, upper and lower case letters are treated as different characters.

2. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.[17] □

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 77 in Section 6.1.

Example 6.5.7. Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation[18] which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

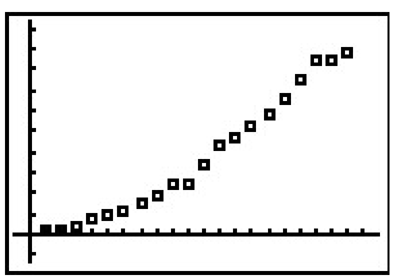
Solution. We set pH = 7.4 and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr. □

Another place logarithms are used is in data analysis. Suppose, for instance, we wish to model the spread of influenza A (H1N1), the so-called 'Swine Flu'. Below is data taken from the World Health Organization (WHO) where t represents the number of days since April 28, 2009, and N represents the number of confirmed cases of H1N1 virus worldwide.

t	1	2	3	4	5	6	7	8	9	10	11	12	13
N	148	257	367	658	898	1085	1490	1893	2371	2500	3440	4379	4694

t	14	15	16	17	18	19	20
N	5251	5728	6497	7520	8451	8480	8829

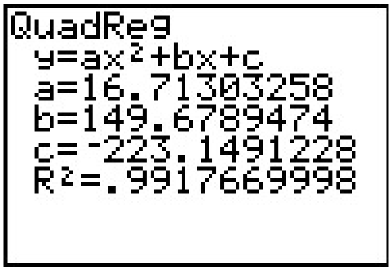
Making a scatter plot of the data treating t as the independent variable and N as the dependent variable gives

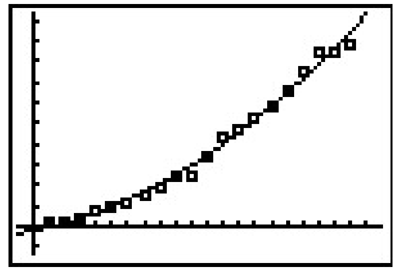


Which models are suggested by the shape of the data? Thinking back Section 2.5, we try a Quadratic Regression, with pretty good results.

[17] Since there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increased.

[18] Derived from the Henderson-Hasselbalch Equation. See Exercise 43 in Section 6.2. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.



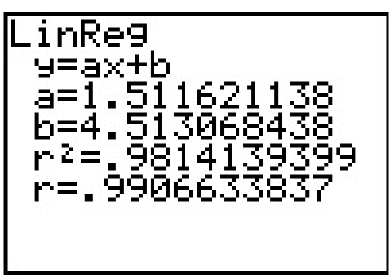


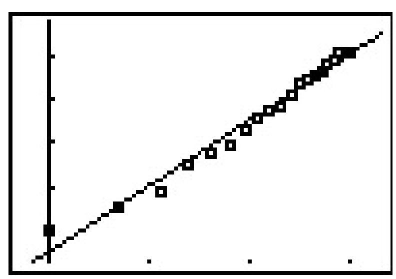
However, is there any scientific reason for the data to be quadratic? Are there other models which fit the data equally well, or better? Scientists often use logarithms in an attempt to 'linearize' data sets - in other words, transform the data sets to produce ones which result in straight lines. To see how this could work, suppose we guessed the relationship between N and t was some kind of power function, not necessarily quadratic, say $N = Bt^A$. To try to determine the A and B, we can take the natural log of both sides and get $\ln(N) = \ln\left(Bt^A\right)$. Using properties of logs to expand the right hand side of this equation, we get $\ln(N) = A\ln(t) + \ln(B)$. If we set $X = \ln(t)$ and $Y = \ln(N)$, this equation becomes $Y = AX + \ln(B)$. In other words, we have a line with slope A and Y-intercept $\ln(B)$. So, instead of plotting N versus t, we plot $\ln(N)$ versus $\ln(t)$.

$\ln(t)$	0	0.693	1.099	1.386	1.609	1.792	1.946	2.079	2.197	2.302	2.398	2.485	2.565
$\ln(N)$	4.997	5.549	5.905	6.489	6.800	6.989	7.306	7.546	7.771	7.824	8.143	8.385	8.454

$\ln(t)$	2.639	2.708	2.773	2.833	2.890	2.944	2.996
$\ln(N)$	8.566	8.653	8.779	8.925	9.042	9.045	9.086

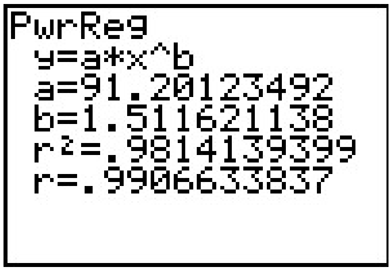
Running a linear regression on the data gives

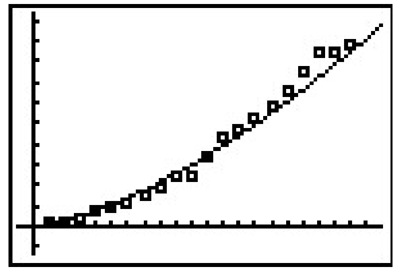




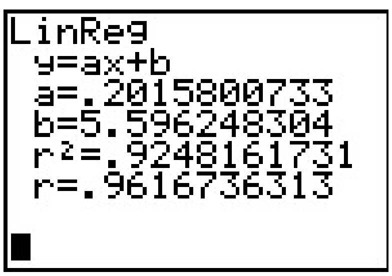
The slope of the regression line is $a \approx 1.512$ which corresponds to our exponent A. The y-intercept $b \approx 4.513$ corresponds to $\ln(B)$, so that $B \approx 91.201$. Hence, we get the model $N = 91.201t^{1.512}$, something from Section 5.3. Of course, the calculator has a built-in 'Power Regression' feature. If we apply this to our original data set, we get the same model we arrived at before.[19]

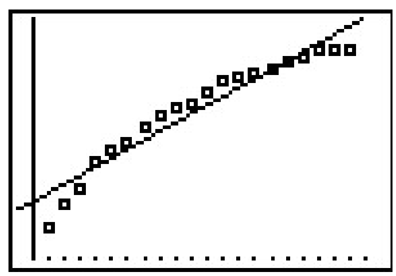
[19] Critics may question why the authors of the book have chosen to even discuss linearization of data when the calculator has a Power Regression built-in and ready to go. Our response: talk to your science faculty.



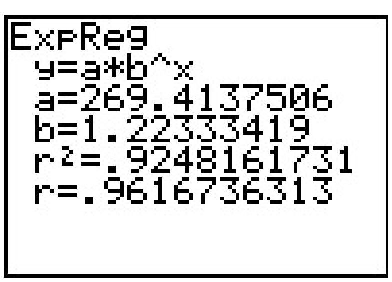


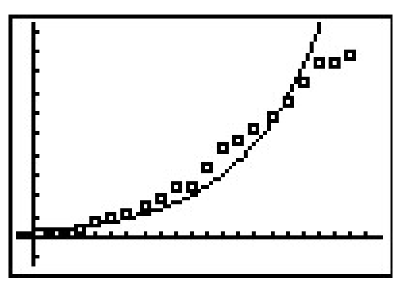
This is all well and good, but the quadratic model appears to fit the data better, and we've yet to mention any scientific principle which would lead us to believe the actual spread of the flu follows any kind of power function at all. If we are to attack this data from a scientific perspective, it does seem to make sense that, at least in the early stages of the outbreak, the more people who have the flu, the faster it will spread, which leads us to proposing an uninhibited growth model. If we assume $N = Be^{At}$ then, taking logs as before, we get $\ln(N) = At + \ln(B)$. If we set $X = t$ and $Y = \ln(N)$, then, once again, we get $Y = AX + \ln(B)$, a line with slope A and Y-intercept $\ln(B)$. Plotting $\ln(N)$ versus t gives the following linear regression.



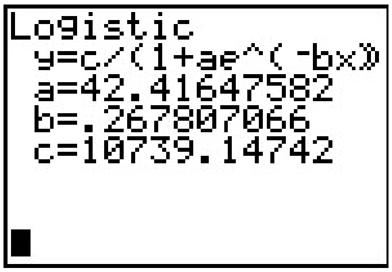


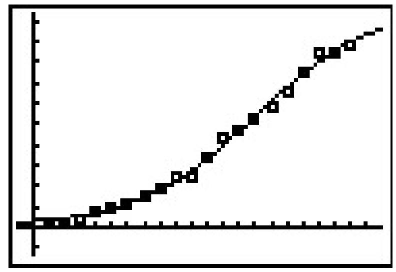
We see the slope is $a \approx 0.202$ and which corresponds to A in our model, and the y-intercept is $b \approx 5.596$ which corresponds to $\ln(B)$. We get $B \approx 269.414$, so that our model is $N = 269.414e^{0.202t}$. Of course, the calculator has a built-in 'Exponential Regression' feature which produces what appears to be a different model $N = 269.414(1.22333419)^t$. Using properties of exponents, we write $e^{0.202t} = \left(e^{0.202}\right)^t \approx (1.223848)^t$, which, had we carried more decimal places, would have matched the base of the calculator model exactly.





The exponential model didn't fit the data as well as the quadratic or power function model, but it stands to reason that, perhaps, the spread of the flu is not unlike that of the spread of a rumor and that a logistic model can be used to model the data. The calculator does have a 'Logistic Regression' feature, and using it produces the model $N = \frac{10739.147}{1+42.416e^{0.268t}}$.





This appears to be an excellent fit, but there is no friendly coefficient of determination, R^2, by which to judge this numerically. There are good reasons for this, but they are far beyond the scope of the text. Which of the models, quadratic, power, exponential, or logistic is the 'best model'? If by 'best' we mean 'fits closest to the data,' then the quadratic and logistic models are arguably the winners with the power function model a close second. However, if we think about the science behind the spread of the flu, the logistic model gets an edge. For one thing, it takes into account that only a finite number of people will ever get the flu (according to our model, 10,739), whereas the quadratic model predicts no limit to the number of cases. As we have stated several times before in the text, mathematical models, regardless of their sophistication, are just that: models, and they all have their limitations.[20]

[20]Speaking of limitations, as of June 3, 2009, there were 19,273 confirmed cases of influenza A (H1N1). This is well above our prediction of 10,739. Each time a new report is issued, the data set increases and the model must be recalculated. We leave this recalculation to the reader.

6.5.3 Exercises

For each of the scenarios given in Exercises 1 - 6,

- Find the amount A in the account as a function of the term of the investment t in years.
- Determine how much is in the account after 5 years, 10 years, 30 years and 35 years. Round your answers to the nearest cent.
- Determine how long will it take for the initial investment to double. Round your answer to the nearest year.
- Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. Round your answer to two decimal places.

1. \$500 is invested in an account which offers 0.75%, compounded monthly.
2. \$500 is invested in an account which offers 0.75%, compounded continuously.
3. \$1000 is invested in an account which offers 1.25%, compounded monthly.
4. \$1000 is invested in an account which offers 1.25%, compounded continuously.
5. \$5000 is invested in an account which offers 2.125%, compounded monthly.
6. \$5000 is invested in an account which offers 2.125%, compounded continuously.
7. Look back at your answers to Exercises 1 - 6. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.
8. How much money needs to be invested now to obtain \$2000 in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.
9. How much money needs to be invested now to obtain \$5000 in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.
10. On May, 31, 2009, the Annual Percentage Rate listed at Jeff's bank for regular savings accounts was 0.25% compounded monthly. Use Equation 6.2 to answer the following.
    (a) If $P = 2000$ what is $A(8)$?
    (b) Solve the equation $A(t) = 4000$ for t.
    (c) What principal P should be invested so that the account balance is \$2000 is three years?

11. Jeff's bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.

    (a) If $P = 2000$ what is $A(8)$?

    (b) Solve the equation $A(t) = 4000$ for t.

    (c) What principal P should be invested so that the account balance is \$2000 in three years?

    (d) The Annual Percentage Yield is the simple interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

12. A finance company offers a promotion on \$5000 loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount would be due?

13. Use Equation 6.2 to show that the time it takes for an investment to double in value does not depend on the principal P, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[21] in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

In Exercises 14 - 18, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and k is the decay constant. For each isotope:

- Find the decay constant k. Round your answer to four decimal places.
- Find a function which gives the amount of isotope A which remains after time t. (Keep the units of A and t the same as the given data.)
- Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

14. Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

15. Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

16. Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

17. Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

18. Uranium 235, used for nuclear power, initial amount 1 kg grams, half-life 704 million years.

[21] Awesome pun!

19. With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises 14 - 18 to decay does not depend on the initial amount of the substance, but rather, on only the decay constant k. Find a formula, in terms of k only, to determine how long it takes for 90% of a radioactive isotope to decay.

20. In Example 6.1.1 in Section 6.1, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use the properties of logs and/or exponents to rewrite the model in the form $V(t) = 25e^{kt}$.

21. The Gross Domestic Product (GDP) of the US (in billions of dollars) t years after the year 2000 can be modeled by:
$$G(t) = 9743.77e^{0.0514t}$$
    (a) Find and interpret $G(0)$.
    (b) According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was \$14,369.1 billion and the 2010 GDP was \$14,657.8 billion.)

22. The diameter D of a tumor, in millimeters, t days after it is detected is given by:
$$D(t) = 15e^{0.0277t}$$
    (a) What was the diameter of the tumor when it was originally detected?
    (b) How long until the diameter of the tumor doubles?

23. Under optimal conditions, the growth of a certain strain of *E. Coli* is modeled by the Law of Uninhibited Growth $N(t) = N_0e^{kt}$ where N_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.
    (a) Find the growth constant k. Round your answer to four decimal places.
    (b) Find a function which gives the number of bacteria $N(t)$ after t minutes.
    (c) How long until there are 9000 bacteria? Round your answer to the nearest minute.

24. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $N(t) = N_0e^{kt}$.
    (a) Find the growth constant k. Round your answer to four decimal places.
    (b) Find a function which gives the number of yeast (in millions) per cc $N(t)$ after t hours.
    (c) What is the doubling time for this strain of yeast?

25. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0 e^{kt}$ which models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

26. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the Village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $N(t) = N_0 e^{kt}$ were $N(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563) What could be some causes for such a vast discrepancy? For more on this, see Exercise 37.

27. The population of Sasquatch in Bigfoot county is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

(a) Find and interpret $P(0)$.

(b) Find the population of Sasquatch in Bigfoot county in 2013. Round your answer to the nearest Sasquatch.

(c) When will the population of Sasquatch in Bigfoot county reach 60? Round your answer to the nearest year.

(d) Find and interpret the end behavior of the graph of $y = P(t)$. Check your answer using a graphing utility.

28. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

(a) Use Equation 6.5 to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

(b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

(c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

(d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

29. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 6.5 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

30. Use Equation 6.5 to show that $k = -\dfrac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.

31. A pork roast[22] was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F. Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 6.6),

    (a) Express the temperature T (in °F) as a function of time t (in minutes).

    (b) Find the time at which the roast would have dropped to 140°F had it not been carved and eaten.

32. In reference to Exercise 44 in Section 5.3, if Fritzy the Fox's speed is the same as Chewbacca the Bunny's speed, Fritzy's pursuit curve is given by

$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

    Use your calculator to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.

33. The current i measured in amps in a certain electronic circuit with a constant impressed voltage of 120 volts is given by $i(t) = 2 - 2e^{-10t}$ where $t \geq 0$ is the number of seconds after the circuit is switched on. Determine the value of i as $t \to \infty$. (This is called the **steady state** current.)

34. If the voltage in the circuit in Exercise 33 above is switched off after 30 seconds, the current is given by the piecewise-defined function

$$i(t) = \begin{cases} 2 - 2e^{-10t} & \text{if } 0 \leq t < 30 \\ \left(2 - 2e^{-300}\right) e^{-10t+300} & \text{if } t \geq 30 \end{cases}$$

    With the help of your calculator, graph $y = i(t)$ and discuss with your classmates the physical significance of the two parts of the graph $0 \leq t < 30$ and $t \geq 30$.

[22]This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

35. In Exercise 26 in Section 2.3, we stated that the cable of a suspension bridge formed a parabola but that a free hanging cable did not. A free hanging cable forms a catenary and its basic shape is given by $y = \frac{1}{2}\left(e^{x} + e^{-x}\right)$. Use your calculator to graph this function. What are its domain and range? What is its end behavior? Is it invertible? How do you think it is related to the function given in Exercise 47 in Section 6.3 and the one given in the answer to Exercise 38 in Section 6.4? When flipped upside down, the catenary makes an arch. The Gateway Arch in St. Louis, Missouri has the shape

$$y = 757.7 - \frac{127.7}{2}\left(e^{\frac{x}{127.7}} + e^{-\frac{x}{127.7}}\right)$$

where x and y are measured in feet and $-315 \leq x \leq 315$. Find the highest point on the arch.

36. In Exercise 6a in Section 2.5, we examined the data set given below which showed how two cats and their surviving offspring can produce over 80 million cats in just ten years. It is virtually impossible to see this data plotted on your calculator, so plot x versus $\ln(x)$ as was done on page 480. Find a linear model for this new data and comment on its goodness of fit. Find an exponential model for the original data and comment on its goodness of fit.

Year x	1	2	3	4	5	6	7	8	9	10
Number of Cats $N(x)$	12	66	382	2201	12680	73041	420715	2423316	13968290	80399780

37. This exercise is a follow-up to Exercise 26 which more thoroughly explores the population growth of Painesville, Ohio. According to Wikipedia, the population of Painesville, Ohio is given by

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	2649	3728	3841	4755	5024	5501	7272	10944	12235	14432

Year t	1960	1970	1980	1990	2000
Population	16116	16536	16351	15699	17503

(a) Use a graphing utility to perform an exponential regression on the data from 1860 through 1920 only, letting $t = 0$ represent the year 1860 as before. How does this calculator model compare with the model you found in Exercise 26? Use the calculator's exponential model to predict the population in 2010. (The 2010 census gave the population as 19,563)

(b) The logistic model fit to *all* of the given data points for the population of Painesville t years after 1860 (again, using $t = 0$ as 1860) is

$$P(t) = \frac{18691}{1 + 9.8505e^{-0.03617t}}$$

According to this model, what should the population of Painesville have been in 2010? (The 2010 census gave the population as 19,563.) What is the population limit of Painesville?

38. According to OhioBiz, the census data for Lake County, Ohio is as follows:

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	15576	15935	16326	18235	21680	22927	28667	41674	50020	75979

Year t	1960	1970	1980	1990	2000
Population	148700	197200	212801	215499	227511

(a) Use your calculator to fit a logistic model to these data, using $x = 0$ to represent the year 1860.

(b) Graph these data and your logistic function on your calculator to judge the reasonableness of the fit.

(c) Use this model to estimate the population of Lake County in 2010. (The 2010 census gave the population to be 230,041.)

(d) According to your model, what is the population limit of Lake County, Ohio?

39. According to facebook, the number of active users of facebook has grown significantly since its initial launch from a Harvard dorm room in February 2004. The chart below has the approximate number $U(x)$ of active users, in millions, x months after February 2004. For example, the first entry $(10, 1)$ means that there were 1 million active users in December 2004 and the last entry $(77, 500)$ means that there were 500 million active users in July 2010.

Month x	10	22	34	38	44	54	59	60	62	65	67	70	72	77
Active Users in Millions $U(x)$	1	5.5	12	20	50	100	150	175	200	250	300	350	400	500

With the help of your classmates, find a model for this data.

40. Each Monday during the registration period before the Fall Semester at LCCC, the Enrollment Planning Council gets a report prepared by the data analysts in Institutional Effectiveness and Planning.[23] While the ongoing enrollment data is analyzed in many different ways, we shall focus only on the overall headcount. Below is a chart of the enrollment data for Fall Semester 2008. It starts 21 weeks before "Opening Day" and ends on "Day 15" of the semester, but we have relabeled the top row to be $x = 1$ through $x = 24$ so that the math is easier. (Thus, $x = 22$ is Opening Day.)

Week x	1	2	3	4	5	6	7	8
Total Headcount	1194	1564	2001	2475	2802	3141	3527	3790

Week x	9	10	11	12	13	14	15	16
Total Headcount	4065	4371	4611	4945	5300	5657	6056	6478

[23] The authors thank Dr. Wendy Marley and her staff for this data and Dr. Marcia Ballinger for the permission to use it in this problem.

Week x	17	18	19	20	21	22	23	24
Total Headcount	7161	7772	8505	9256	10201	10743	11102	11181

With the help of your classmates, find a model for this data. Unlike most of the phenomena we have studied in this section, there is no single differential equation which governs the enrollment growth. Thus there is no scientific reason to rely on a logistic function even though the data plot may lead us to that model. What are some factors which influence enrollment at a community college and how can you take those into account mathematically?

41. When we wrote this exercise, the Enrollment Planning Report for Fall Semester 2009 had only 10 data points for the first 10 weeks of the registration period. Those numbers are given below.

Week x	1	2	3	4	5	6	7	8	9	10
Total Headcount	1380	2000	2639	3153	3499	3831	4283	4742	5123	5398

With the help of your classmates, find a model for this data and make a prediction for the Opening Day enrollment as well as the Day 15 enrollment. (WARNING: The registration period for 2009 was one week shorter than it was in 2008 so Opening Day would be $x = 21$ and Day 15 is $x = 23$.)

6.5.4 Answers

1. - $A(t) = 500\left(1 + \frac{0.0075}{12}\right)^{12t}$
   - $A(5) \approx \$519.10$, $A(10) \approx \$538.93$, $A(30) \approx \$626.12$, $A(35) \approx \$650.03$
   - It will take approximately 92 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.85. This means that the investment is growing at an average rate of \$4.85 per year at this point.

2. - $A(t) = 500e^{0.0075t}$
   - $A(5) \approx \$519.11$, $A(10) \approx \$538.94$, $A(30) \approx \$626.16$, $A(35) \approx \$650.09$
   - It will take approximately 92 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.86. This means that the investment is growing at an average rate of \$4.86 per year at this point.

3. - $A(t) = 1000\left(1 + \frac{0.0125}{12}\right)^{12t}$
   - $A(5) \approx \$1064.46$, $A(10) \approx \$1133.07$, $A(30) \approx \$1454.71$, $A(35) \approx \$1548.48$
   - It will take approximately 55 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.23. This means that the investment is growing at an average rate of \$19.23 per year at this point.

4. - $A(t) = 1000e^{0.0125t}$
   - $A(5) \approx \$1064.49$, $A(10) \approx \$1133.15$, $A(30) \approx \$1454.99$, $A(35) \approx \$1548.83$
   - It will take approximately 55 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.24. This means that the investment is growing at an average rate of \$19.24 per year at this point.

5. - $A(t) = 5000\left(1 + \frac{0.02125}{12}\right)^{12t}$
   - $A(5) \approx \$5559.98$, $A(10) \approx \$6182.67$, $A(30) \approx \$9453.40$, $A(35) \approx \$10512.13$
   - It will take approximately 33 years for the investment to double.

- The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.80. This means that the investment is growing at an average rate of \$116.80 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 220.83. This means that the investment is growing at an average rate of \$220.83 per year at this point.

6.
- $A(t) = 5000e^{0.02125t}$
- $A(5) \approx \$5560.50$, $A(10) \approx \$6183.83$, $A(30) \approx \$9458.73$, $A(35) \approx \$10519.05$
- It will take approximately 33 years for the investment to double.
- The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.91. This means that the investment is growing at an average rate of \$116.91 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 221.17. This means that the investment is growing at an average rate of \$221.17 per year at this point.

8. $P = \frac{2000}{e^{0.0025 \cdot 3}} \approx \$1985.06$

9. $P = \frac{5000}{\left(1+\frac{0.0225}{12}\right)^{12 \cdot 10}} \approx \$3993.42$

10. (a) $A(8) = 2000\left(1 + \frac{0.0025}{12}\right)^{12 \cdot 8} \approx \$2040.40$

(b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0025}{12}\right)} \approx 277.29$ years

(c) $P = \dfrac{2000}{\left(1 + \frac{0.0025}{12}\right)^{36}} \approx \$1985.06$

11. (a) $A(8) = 2000\left(1 + \frac{0.0225}{12}\right)^{12 \cdot 8} \approx \$2394.03$

(b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0225}{12}\right)} \approx 30.83$ years

(c) $P = \dfrac{2000}{\left(1 + \frac{0.0225}{12}\right)^{36}} \approx \$1869.57$

(d) $\left(1 + \frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%

12. $A(3) = 5000e^{0.299 \cdot 3} \approx \$12,226.18$, $A(6) = 5000e^{0.299 \cdot 6} \approx \$30,067.29$

14.
- $k = \frac{\ln(1/2)}{5.27} \approx -0.1315$
- $A(t) = 50e^{-0.1315t}$
- $t = \frac{\ln(0.1)}{-0.1315} \approx 17.51$ years.

15.
- $k = \frac{\ln(1/2)}{14} \approx -0.0495$
- $A(t) = 2e^{-0.0495t}$
- $t = \frac{\ln(0.1)}{-0.0495} \approx 46.52$ days.

16.
- $k = \frac{\ln(1/2)}{27.7} \approx -0.0250$
- $A(t) = 75e^{-0.0250t}$
- $t = \frac{\ln(0.1)}{-0.025} \approx 92.10$ days.

17.
- $k = \frac{\ln(1/2)}{432.7} \approx -0.0016$
- $A(t) = 0.29e^{-0.0016t}$
- $t = \frac{\ln(0.1)}{-0.0016} \approx 1439.11$ years.

18.
- $k = \frac{\ln(1/2)}{704} \approx -0.0010$
- $A(t) = e^{-0.0010t}$
- $t = \frac{\ln(0.1)}{-0.0010} \approx 2302.58$ million years, or 2.30 billion years.

19. $t = \frac{\ln(0.1)}{k} = -\frac{\ln(10)}{k}$

20. $V(t) = 25e^{\ln\left(\frac{4}{5}\right)t} \approx 25e^{-0.22314355t}$

21. (a) $G(0) = 9743.77$ This means that the GDP of the US in 2000 was \$9743.77 billion dollars.
    (b) $G(7) = 13963.24$ and $G(10) = 16291.25$, so the model predicted a GDP of \$13,963.24 billion in 2007 and \$16,291.25 billion in 2010.

22. (a) $D(0) = 15$, so the tumor was 15 millimeters in diameter when it was first detected.
    (b) $t = \frac{\ln(2)}{0.0277} \approx 25$ days.

23. (a) $k = \frac{\ln(2)}{20} \approx 0.0346$
    (b) $N(t) = 1000e^{0.0346t}$
    (c) $t = \frac{\ln(9)}{0.0346} \approx 63$ minutes

24. (a) $k = \frac{1}{2}\frac{\ln(6)}{2.5} \approx 0.4377$
    (b) $N(t) = 2.5e^{0.4377t}$
    (c) $t = \frac{\ln(2)}{0.4377} \approx 1.58$ hours

25. $N_0 = 52$, $k = \frac{1}{3}\ln\left(\frac{118}{52}\right) \approx 0.2731$, $N(t) = 52e^{0.2731t}$. $N(6) \approx 268$.

26. $N_0 = 2649$, $k = \frac{1}{60}\ln\left(\frac{7272}{2649}\right) \approx 0.0168$, $N(t) = 2649e^{0.0168t}$. $N(150) \approx 32923$, so the population of Painesville in 2010 based on this model would have been 32,923.

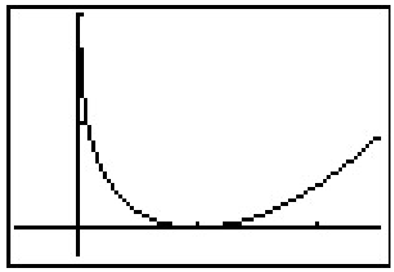
27. (a) $P(0) = \frac{120}{4.167} \approx 29$. There are 29 Sasquatch in Bigfoot County in 2010.
    (b) $P(3) = \frac{120}{1+3.167e^{-0.05(3)}} \approx 32$ Sasquatch.
    (c) $t = 20\ln(3.167) \approx 23$ years.
    (d) As $t \to \infty$, $P(t) \to 120$. As time goes by, the Sasquatch Population in Bigfoot County will approach 120. Graphically, $y = P(x)$ has a horizontal asymptote $y = 120$.

28. (a) $A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$
    (b) $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains
    (c) $t \approx \dfrac{\ln(.42)}{-0.00012097} \approx 7171$ years old

29. $A(t) = 2.3e^{-0.0138629t}$

31. (a) $T(t) = 75 + 105e^{-0.005005t}$
    (b) The roast would have cooled to 140°F in about 95 minutes.

32. From the graph, it appears that as $x \to 0^+$, $y \to \infty$. This is due to the presence of the $\ln(x)$ term in the function. This means that Fritzy will never catch Chewbacca, which makes sense since Chewbacca has a head start and Fritzy only runs as fast as he does.



$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$

33. The steady state current is 2 amps.

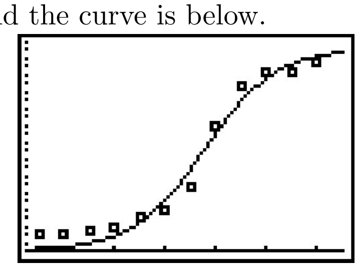
36. The linear regression on the data below is $y = 1.74899x + 0.70739$ with $r^2 \approx 0.999995$. This is an excellent fit.

x	1	2	3	4	5	6	7	8	9	10
$\ln(N(x))$	2.4849	4.1897	5.9454	7.6967	9.4478	11.1988	12.9497	14.7006	16.4523	18.2025

$N(x) = 2.02869(5.74879)^x = 2.02869e^{1.74899x}$ with $r^2 \approx 0.999995$. This is also an excellent fit and corresponds to our linearized model because $\ln(2.02869) \approx 0.70739$.

37. (a) The calculator gives: $y = 2895.06(1.0147)^x$. Graphing this along with our answer from Exercise 26 over the interval $[0, 60]$ shows that they are pretty close. From this model, $y(150) \approx 25840$ which once again overshoots the actual data value.
    (b) $P(150) \approx 18717$, so this model predicts 17,914 people in Painesville in 2010, a more conservative number than was recorded in the 2010 census. As $t \to \infty$, $P(t) \to 18691$. So the limiting population of Painesville based on this model is 18,691 people.

38. (a) $y = \dfrac{242526}{1 + 874.62e^{-0.07113x}}$, where x is the number of years since 1860.
    (b) The plot of the data and the curve is below.



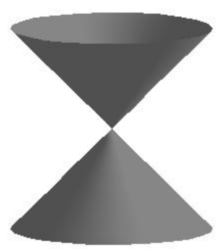
    (c) $y(140) \approx 232889$, so this model predicts 232,889 people in Lake County in 2010.
    (d) As $x \to \infty$, $y \to 242526$, so the limiting population of Lake County based on this model is 242,526 people.

Chapter 7

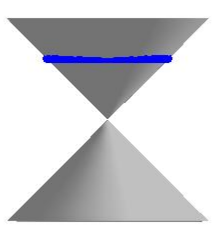
Hooked on Conics

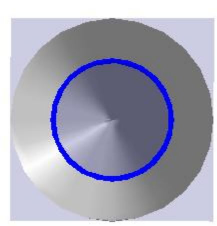
7.1 Introduction to Conics

In this chapter, we study the **Conic Sections** - literally 'sections of a cone'. Imagine a double-napped cone as seen below being 'sliced' by a plane.

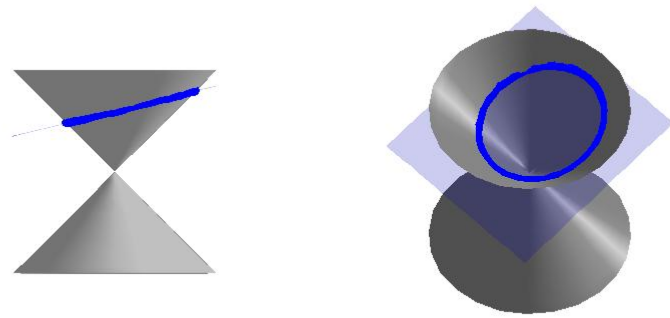


If we slice the cone with a horizontal plane the resulting curve is a **circle**.

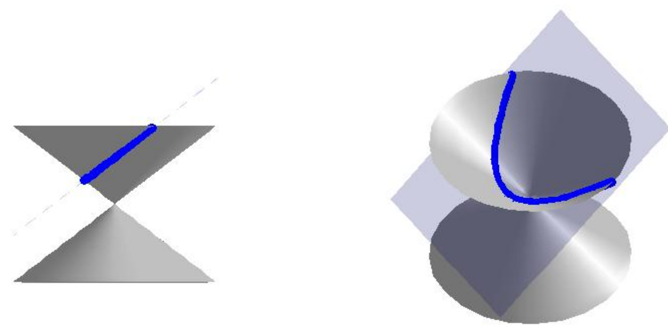




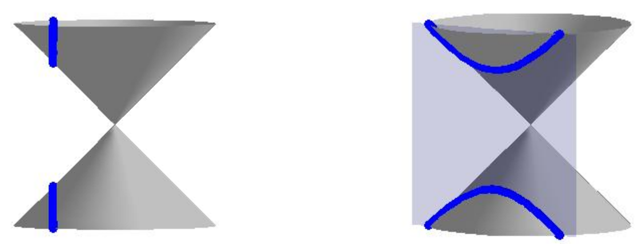
Tilting the plane ever so slightly produces an **ellipse**.



If the plane cuts parallel to the cone, we get a **parabola**.

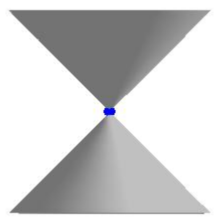


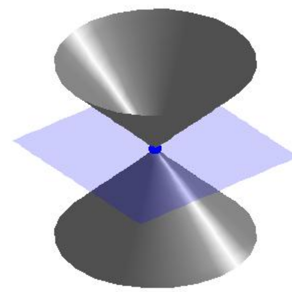
If we slice the cone with a vertical plane, we get a **hyperbola**.

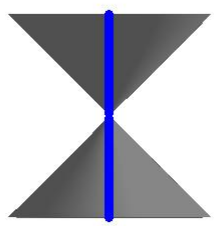


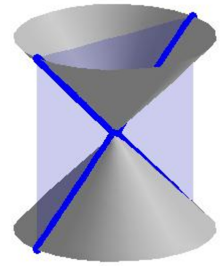
For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's [Mathematics Animated Website](#).

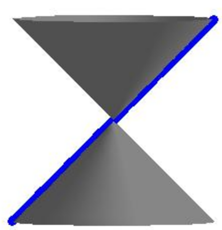
If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

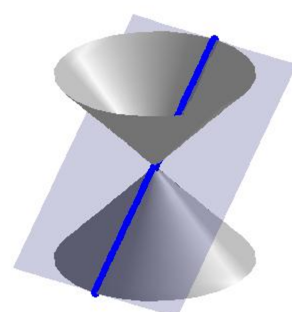










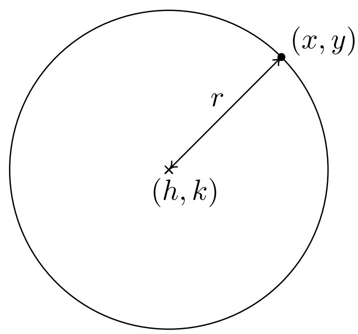


We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. To determine equations which describe these curves, we will make use of their definitions in terms of distances.

7.2 CIRCLES

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 7.1. A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the picture, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

Equation 7.1. The Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

Example 7.2.1. Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

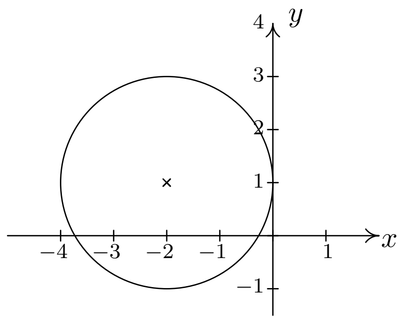
Solution. Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

□

Example 7.2.2. Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Solution. From the standard form of a circle, Equation 7.1, we have that $x+2$ is $x-h$, so $h = -2$ and $y-1$ is $y-k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2 + (y-1)^2 = 4$, we'd be contending with $x^2 + 4x + y^2 - 2y + 1 = 0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 7.1 by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.

2. Complete the square on both variables as needed.

3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 7.2.3. Complete the square to find the center and radius of $3x^2 - 6x + 3y^2 + 4y - 4 = 0$.

Solution.

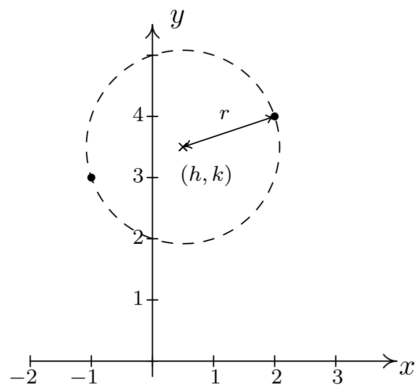
$$\begin{array}{rcll}
3x^2 - 6x + 3y^2 + 4y - 4 & = & 0 & \\
3x^2 - 6x + 3y^2 + 4y & = & 4 & \text{add 4 to both sides} \\
3\left(x^2 - 2x\right) + 3\left(y^2 + \frac{4}{3}y\right) & = & 4 & \text{factor out leading coefficients} \\
3\left(x^2 - 2x + \underline{1}\right) + 3\left(y^2 + \frac{4}{3}y + \underline{\underline{\frac{4}{9}}}\right) & = & 4 + 3\underline{(1)} + 3\underline{\underline{\left(\frac{4}{9}\right)}} & \text{complete the square in } x,\ y \\
3(x-1)^2 + 3\left(y + \frac{2}{3}\right)^2 & = & \frac{25}{3} & \text{factor} \\
(x-1)^2 + \left(y + \frac{2}{3}\right)^2 & = & \frac{25}{9} & \text{divide both sides by 3}
\end{array}$$

From Equation 7.1, we identify $x - 1$ as $x - h$, so $h = 1$, and $y + \frac{2}{3}$ as $y - k$, so $k = -\frac{2}{3}$. Hence, the center is $(h, k) = \left(1, -\frac{2}{3}\right)$. Furthermore, we see that $r^2 = \frac{25}{9}$ so the radius is $r = \frac{5}{3}$. □

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

Example 7.2.4. Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$ $\square$

We close this section with the most important[1] circle in all of mathematics: the **Unit Circle**.

Definition 7.2. The **Unit Circle** is the circle centered at $(0, 0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2 + y^2 = 1$.

Example 7.2.5. Find the points on the unit circle with y-coordinate $\dfrac{\sqrt{3}}{2}$.

Solution. We replace y with $\dfrac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to get

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4} + x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$. $\square$

[1] While this may seem like an opinion, it is indeed a fact. See Chapters 10 and 11 for details.

7.2.1 Exercises

In Exercises 1 - 6, find the standard equation of the circle and then graph it.

1. Center $(-1, -5)$, radius 10
2. Center $(4, -2)$, radius 3
3. Center $\left(-3, \frac{7}{13}\right)$, radius $\frac{1}{2}$
4. Center $(5, -9)$, radius $\ln(8)$
5. Center $\left(-e, \sqrt{2}\right)$, radius π
6. Center (π, e^2), radius $\sqrt[3]{91}$

In Exercises 7 - 12, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

7. $x^2 - 4x + y^2 + 10y = -25$
8. $-2x^2 - 36x - 2y^2 - 112 = 0$
9. $x^2 + y^2 + 8x - 10y - 1 = 0$
10. $x^2 + y^2 + 5x - y - 1 = 0$
11. $4x^2 + 4y^2 - 24y + 36 = 0$
12. $x^2 + x + y^2 - \frac{6}{5}y = 1$

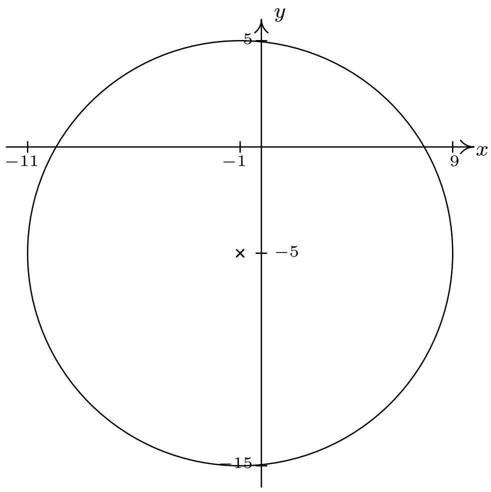
In Exercises 13 - 16, find the standard equation of the circle which satisfies the given criteria.

13. center $(3, 5)$, passes through $(-1, -2)$
14. center $(3, 6)$, passes through $(-1, 4)$
15. endpoints of a diameter: $(3, 6)$ and $(-1, 4)$
16. endpoints of a diameter: $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$
17. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[2] Find an equation for the wheel assuming that its center lies on the y-axis and that the ground is the x-axis.
18. Verify that the following points lie on the Unit Circle: $(\pm 1, 0)$, $(0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$
19. Discuss with your classmates how to obtain the standard equation of a circle, Equation 7.1, from the equation of the Unit Circle, $x^2 + y^2 = 1$ using the transformations discussed in Section 1.7. (Thus every circle is just a few transformations away from the Unit Circle.)
20. Find an equation for the function represented graphically by the top half of the Unit Circle. Explain how the transformations is Section 1.7 can be used to produce a function whose graph is either the top or bottom of an arbitrary circle.
21. Find a one-to-one function whose graph is half of a circle. (Hint: Think piecewise.)

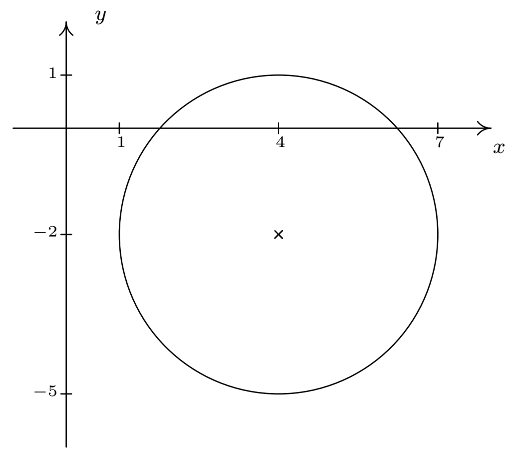
[2]Source: Cedar Point's webpage.

7.2.2 Answers

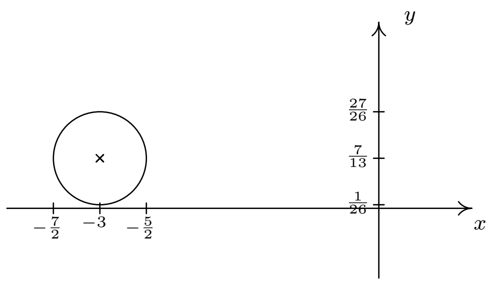
1. $(x+1)^2 + (y+5)^2 = 100$



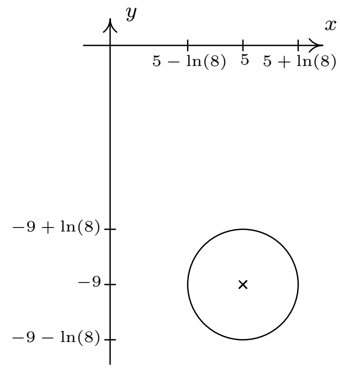
2. $(x-4)^2 + (y+2)^2 = 9$



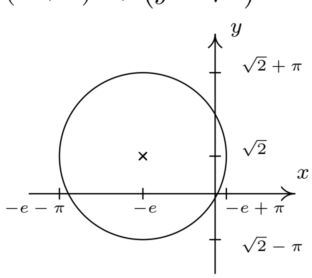
3. $(x+3)^2 + \left(y - \frac{7}{13}\right)^2 = \frac{1}{4}$



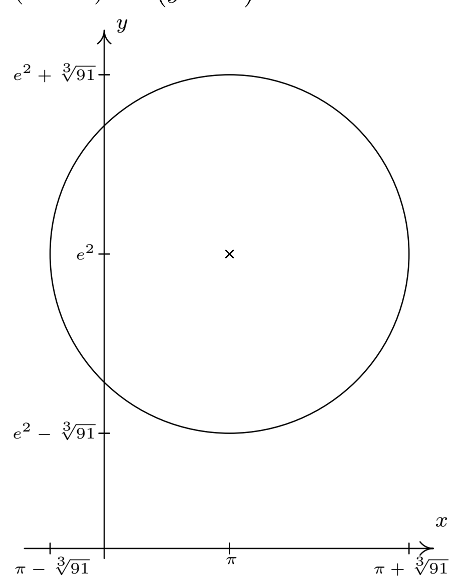
4. $(x-5)^2 + (y+9)^2 = (\ln(8))^2$



5. $(x+e)^2 + \left(y - \sqrt{2}\right)^2 = \pi^2$



6. $(x-\pi)^2 + \left(y - e^2\right)^2 = 91^{\frac{2}{3}}$



7. $(x-2)^2+(y+5)^2=4$
   Center $(2,-5)$, radius $r=2$

8. $(x+9)^2+y^2=25$
   Center $(-9,0)$, radius $r=5$

9. $(x+4)^2+(y-5)^2=42$
   Center $(-4,5)$, radius $r=\sqrt{42}$

10. $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
    Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$

11. $x^2+(y-3)^2=0$
    This is not a circle.

12. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
    Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

13. $(x-3)^2+(y-5)^2=65$

14. $(x-3)^2+(y-6)^2=20$

15. $(x-1)^2+(y-5)^2=5$

16. $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{2}$

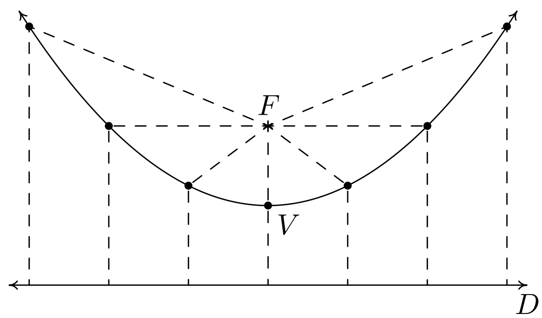
17. $x^2+(y-72)^2=4096$

7.3 Parabolas

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a **parabola**. To our surprise and delight, we may also define parabolas in terms of distance.

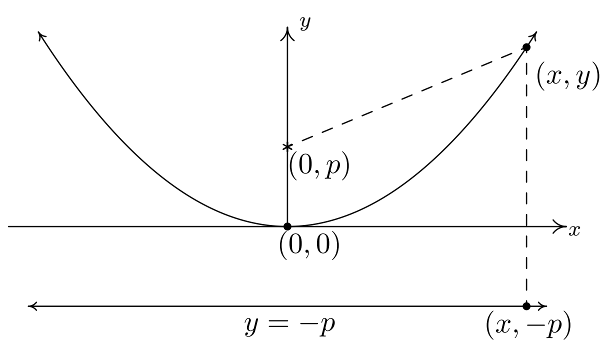
Definition 7.3. Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section 2.3. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes



From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x, -p)$ to (x, y). Using the Distance Formula, Equation 1.1, we get

[1] We'll talk more about what 'directed' means later.

$$\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} & \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} & \\
x^2+(y-p)^2 &= (y+p)^2 & \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 & \text{expand quantities} \\
x^2 &= 4py & \text{gather like terms}
\end{aligned}$$

Solving for y yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation 2.4 with $a = \frac{1}{4p}$ and vertex $(0,0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

If we choose to place the vertex at an arbitrary point (h,k), we arrive at the following formula using either transformations from Section 1.7 or re-deriving the formula from Definition 7.3.

Equation 7.2. The Standard Equation of a Vertical[a] Parabola: The equation of a (vertical) parabola with vertex (h,k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

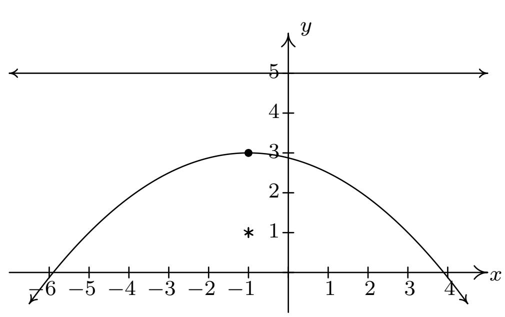
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a]That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

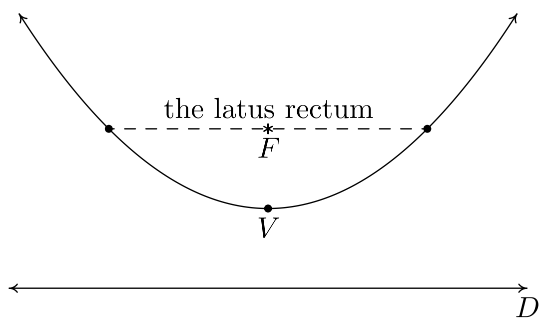
Example 7.3.1. Graph $(x+1)^2 = -8(y-3)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.2. Here, $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 3$ so $k = 3$. Hence, the vertex is $(-1, 3)$. We also see that $4p = -8$ so $p = -2$. Since $p < 0$, the focus will be below the vertex and the parabola will open downwards.



The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. From $(-1, 3)$, we move down 2 units and find the focus at $(-1, 1)$. The directrix, then, is 2 units above the vertex, so it is the line $y = 5$. □

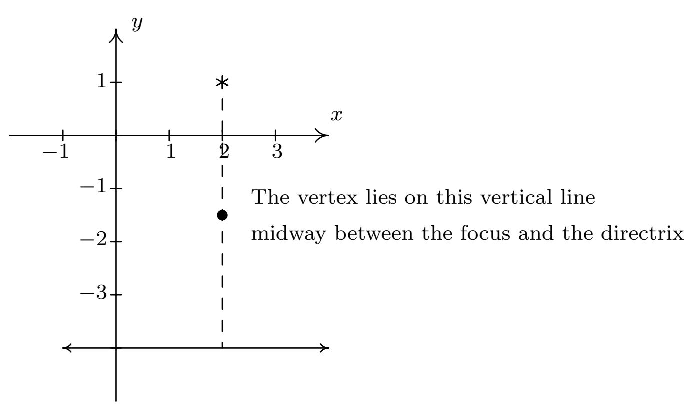
Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[2] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.



It turns out[3] that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p|$, which, in light of Equation 7.2, is easy to find. In our last example, for instance, when graphing $(x + 1)^2 = -8(y - 3)$, we can use the fact that the focal diameter is $|-8| = 8$, which means the parabola is 8 units wide at the focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

Example 7.3.2. Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

Solution. Sketching the data yields,



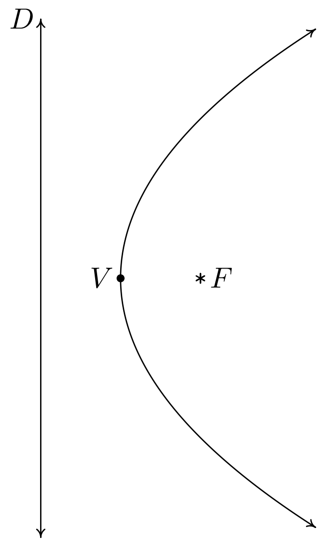
[2] No, I'm not making this up.
[3] Consider this an exercise to show what follows.

From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an x-coordinate of 2. To find the y-coordinate, we note that the distance from the focus to the directrix is $1-(-4)=5$, which means the vertex lies $\frac{5}{2}$ units (halfway) below the focus. Starting at $(2,1)$ and moving down $5/2$ units leaves us at $(2,-3/2)$, which is our vertex. Since the parabola opens upwards, we know p is positive. Thus $p=5/2$. Plugging all of this data into Equation 7.2 give us

$$\begin{aligned}(x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y-\left(-\frac{3}{2}\right)\right)\\(x-2)^2 &= 10\left(y+\frac{3}{2}\right)\end{aligned}$$

$\square$

If we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[4] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Equation 7.3. The Standard Equation of a Horizontal Parabola: The equation of a (horizontal) parabola with vertex (h,k) and focal length $|p|$ is

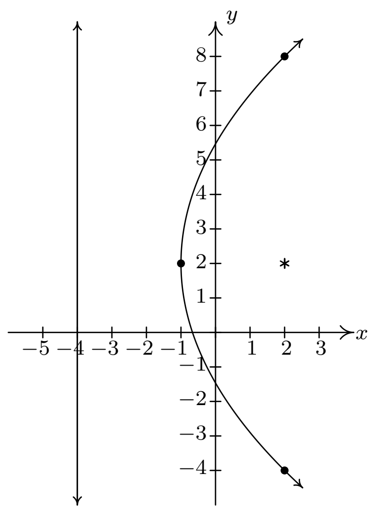
$$(y-k)^2 = 4p(x-h)$$

If $p>0$, the parabola opens to the right; if $p<0$, it opens to the left.

[4]plural of 'directrix'

Example 7.3.3. Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 7.3. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, the vertex is $(-1,2)$. We also see that $4p=12$ so $p=3$. Since $p>0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p|=3$, which means the focus is 3 units to the right. If we start at $(-1,2)$ and move right 3 units, we arrive at the focus $(2,2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1,2)$, we'd be on the vertical line $x=-4$. Since the focal diameter is $|4p|=12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.



□

As with circles, not all parabolas will come to us in the forms in Equations 7.2 or 7.3. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Write the Equation of a Parabola in Standard Form

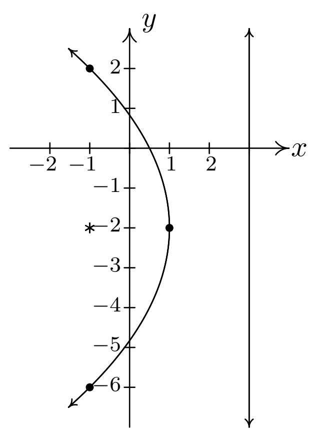
1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

Example 7.3.4. Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

Solution. We need a perfect square (in this case, using y) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the x) on the other.

$$\begin{aligned} y^2+4y+8x &= 4 \\ y^2+4y &= -8x+4 \\ y^2+4y+4 &= -8x+4+4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x+8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

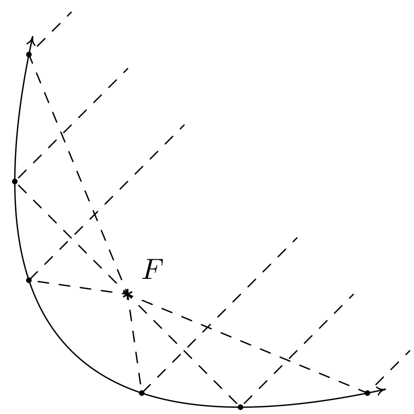
Now that the equation is in the form given in Equation 7.3, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+2$ so $k=-2$. Hence, the vertex is $(1,-2)$. We also see that $4p=-8$ so that $p=-2$. Since $p<0$, the focus will be the left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p|=2$, which means the focus is 2 units to the left of 1, so if we start at $(1,-2)$ and move left 2 units, we arrive at the focus $(-1,-2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1,-2)$, we'd be on the vertical line $x=3$. Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.



□

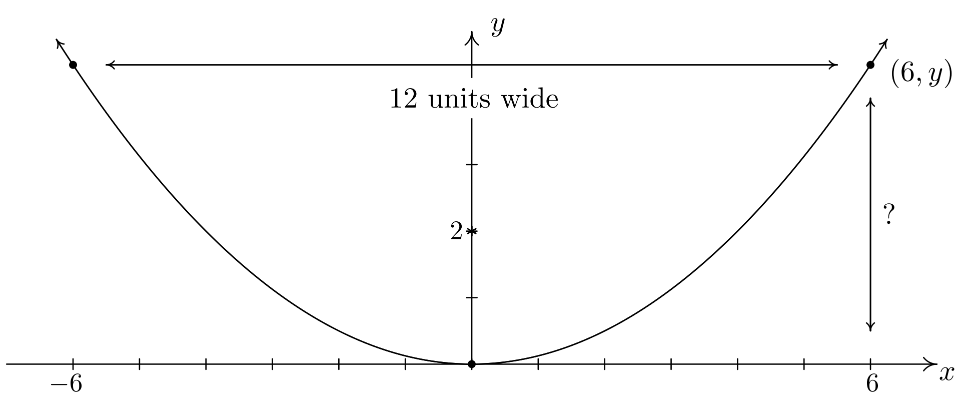
In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.

Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.



Example 7.3.5. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0,0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep. □

7.3.1 Exercises

In Exercises 1 - 8, sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

1. $(x-3)^2 = -16y$
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
3. $(y-2)^2 = -12(x+3)$
4. $(y+4)^2 = 4x$
5. $(x-1)^2 = 4(y+3)$
6. $(x+2)^2 = -20(y-5)$
7. $(y-4)^2 = 18(x-2)$
8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 9 - 14, put the equation into standard form and identify the vertex, focus and directrix.

9. $y^2 - 10y - 27x + 133 = 0$
10. $25x^2 + 20x + 5y - 1 = 0$
11. $x^2 + 2x - 8y + 49 = 0$
12. $2y^2 + 4y + x - 8 = 0$
13. $x^2 - 10x + 12y + 1 = 0$
14. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

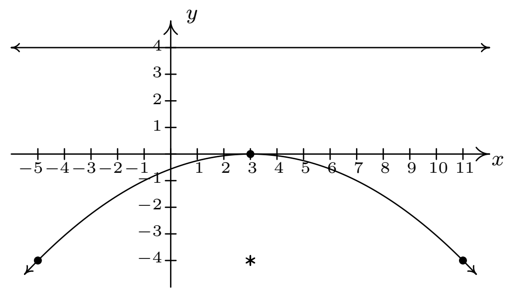
In Exercises 15 - 18, find an equation for the parabola which fits the given criteria.

15. Vertex $(7, 0)$, focus $(0, 0)$
16. Focus $(10, 1)$, directrix $x = 5$
17. Vertex $(-8, -9)$; $(0, 0)$ and $(-16, 0)$ are points on the curve
18. The endpoints of latus rectum are $(-2, -7)$ and $(4, -7)$
19. The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?
20. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape.[5] If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?
21. A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.
22. A popular novelty item is the 'mirage bowl.' Follow this link to see another startling application of the reflective property of the parabola.
23. With the help of your classmates, research spinning liquid mirrors. To get you started, check out this website.

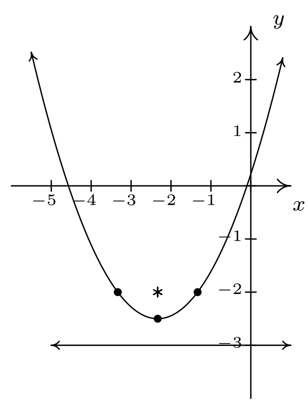
[5]This shape is called a 'parabolic cylinder.'

7.3.2 Answers

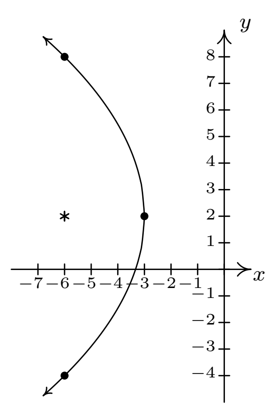
1. $(x-3)^2 = -16y$
   Vertex $(3, 0)$
   Focus $(3, -4)$
   Directrix $y = 4$
   Endpoints of latus rectum $(-5, -4)$, $(11, -4)$



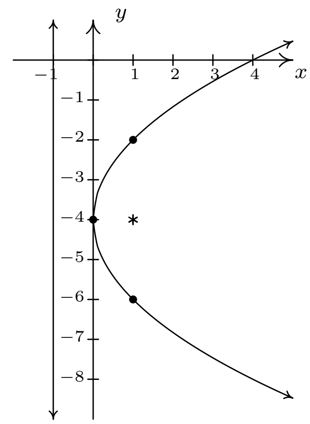
2. $\left(x + \frac{7}{3}\right)^2 = 2\left(y + \frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3}, -\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3}, -2\right)$
   Directrix $y = -3$
   Endpoints of latus rectum $\left(-\frac{10}{3}, -2\right)$, $\left(-\frac{4}{3}, -2\right)$



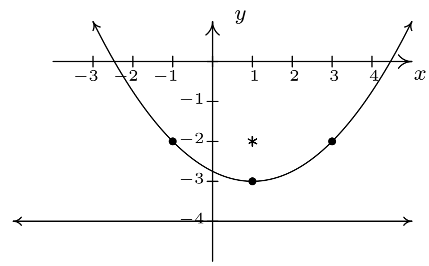
3. $(y-2)^2 = -12(x+3)$
   Vertex $(-3, 2)$
   Focus $(-6, 2)$
   Directrix $x = 0$
   Endpoints of latus rectum $(-6, 8)$, $(-6, -4)$



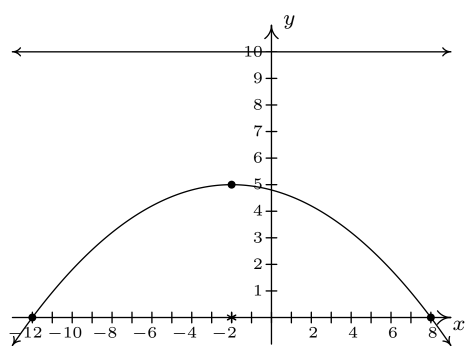
4. $(y+4)^2 = 4x$
Vertex $(0,-4)$
Focus $(1,-4)$
Directrix $x=-1$
Endpoints of latus rectum $(1,-2)$, $(1,-6)$



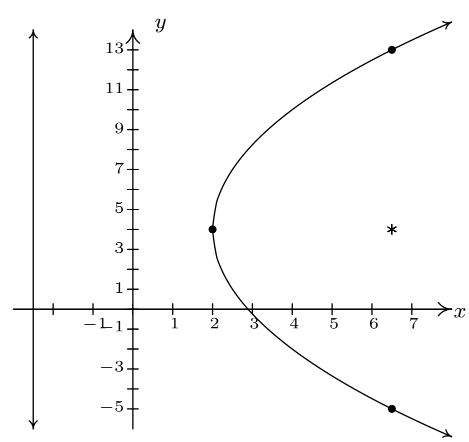
5. $(x-1)^2 = 4(y+3)$
Vertex $(1,-3)$
Focus $(1,-2)$
Directrix $y=-4$
Endpoints of latus rectum $(3,-2)$, $(-1,-2)$



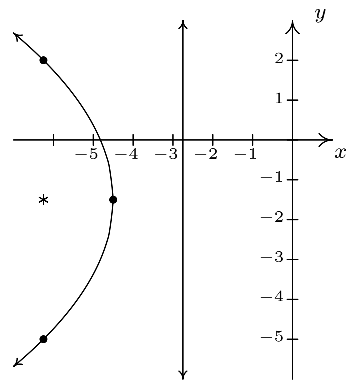
6. $(x+2)^2 = -20(y-5)$
Vertex $(-2,5)$
Focus $(-2,0)$
Directrix $y=10$
Endpoints of latus rectum $(-12,0)$, $(8,0)$



7. $(y-4)^2 = 18(x-2)$
Vertex $(2,4)$
Focus $\left(\frac{13}{2},4\right)$
Directrix $x=-\frac{5}{2}$
Endpoints of latus rectum $\left(\frac{13}{2},-5\right)$, $\left(\frac{13}{2},13\right)$



8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$
Vertex $\left(-\frac{9}{2}, -\frac{3}{2}\right)$
Focus $\left(-\frac{25}{4}, -\frac{3}{2}\right)$
Directrix $x = -\frac{11}{4}$
Endpoints of latus rectum $\left(-\frac{25}{4}, 2\right)$, $\left(-\frac{25}{4}, -5\right)$



9. $(y-5)^2 = 27(x-4)$
Vertex $(4, 5)$
Focus $\left(\frac{43}{4}, 5\right)$
Directrix $x = -\frac{11}{4}$

10. $\left(x+\frac{2}{5}\right)^2 = -\frac{1}{5}(y-1)$
Vertex $\left(-\frac{2}{5}, 1\right)$
Focus $\left(-\frac{2}{5}, \frac{19}{20}\right)$
Directrix $y = \frac{21}{20}$

11. $(x+1)^2 = 8(y-6)$
Vertex $(-1, 6)$
Focus $(-1, 8)$
Directrix $y = 4$

12. $(y+1)^2 = -\frac{1}{2}(x-10)$
Vertex $(10, -1)$
Focus $\left(\frac{79}{8}, -1\right)$
Directrix $x = \frac{81}{8}$

13. $(x-5)^2 = -12(y-2)$
Vertex $(5, 2)$
Focus $(5, -1)$
Directrix $y = 5$

14. $\left(y-\frac{9}{2}\right)^2 = -\frac{4}{3}(x-2)$
Vertex $\left(2, \frac{9}{2}\right)$
Focus $\left(\frac{5}{3}, \frac{9}{2}\right)$
Directrix $x = \frac{7}{3}$

15. $y^2 = -28(x-7)$

16. $(y-1)^2 = 10\left(x-\frac{15}{2}\right)$

17. $(x+8)^2 = \frac{64}{9}(y+9)$

18. $(x-1)^2 = 6\left(y+\frac{17}{2}\right)$ or
$(x-1)^2 = -6\left(y+\frac{11}{2}\right)$

19. The bulb should be placed 0.625 centimeters above the vertex of the mirror. (As verified by Carl himself!)

20. The receiver should be placed 5.0625 centimeters from the vertex of the cross section of the antenna.

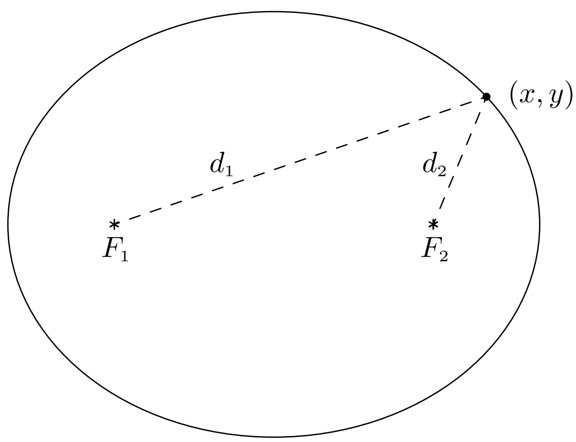
21. The arch can be modeled by $x^2 = -(y-9)$ or $y = 9 - x^2$. One foot in from the base of the arch corresponds to either $x = \pm 2$, so the height is $y = 9 - (\pm 2)^2 = 5$ feet.

7.4 ELLIPSES

In the definition of a circle, Definition 7.1, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 7.4. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

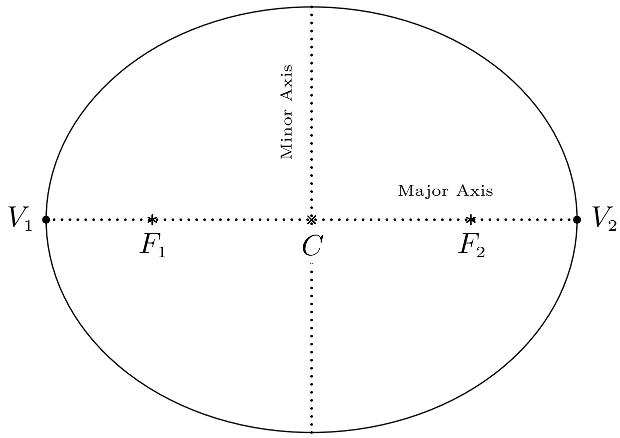
[a] the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

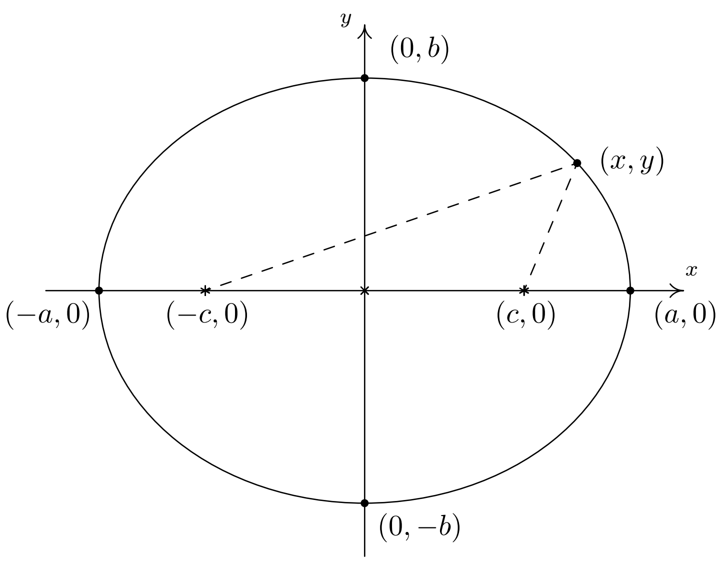
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, and has foci $(c,0)$ and $(-c,0)$ and vertices $(-a,0)$ and $(a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$ (We assume a, b, and c are all positive numbers.) Schematically,



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 7.4. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\
(a+c)+(a-c) &= d \\
2a &= d
\end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0,b)$ is on the ellipse, along with the fact that $d=2a$ to get

$$\begin{array}{rcl}
\text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) & = & 2a \\
\sqrt{(0-(-c))^2+(b-0)^2} + \sqrt{(0-c)^2+(b-0)^2} & = & 2a \\
\sqrt{b^2+c^2} + \sqrt{b^2+c^2} & = & 2a \\
2\sqrt{b^2+c^2} & = & 2a \\
\sqrt{b^2+c^2} & = & a
\end{array}$$

From this, we get $a^2=b^2+c^2$, or $b^2=a^2-c^2$, which will prove useful later. Now consider a point (x,y) on the ellipse. Applying Definition 7.4, we get

$$\begin{array}{rcl}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) & = & 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} & = & 2a \\
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} & = & 2a
\end{array}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{array}{rcl}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} & = & 2a \\
\sqrt{(x+c)^2+y^2} & = & 2a - \sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 & = & \left(2a - \sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 & = & 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \\
4a\sqrt{(x-c)^2+y^2} & = & 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} & = & 4a^2 - 4cx \\
a\sqrt{(x-c)^2+y^2} & = & a^2 - cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 & = & \left(a^2-cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) & = & a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 & = & a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 & = & a^4 - a^2c^2 \\
\left(a^2-c^2\right)x^2 + a^2y^2 & = & a^2\left(a^2-c^2\right)
\end{array}$$

We are nearly finished. Recall that $b^2=a^2-c^2$ so that

$$\begin{array}{rcl}
\left(a^2-c^2\right)x^2 + a^2y^2 & = & a^2\left(a^2-c^2\right) \\
b^2x^2 + a^2y^2 & = & a^2b^2 \\
\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} & = & 1
\end{array}$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h,k), we could use the transformations from Section 1.7 or re-derive the equation using Definition 7.4 and the distance formula to obtain the formula below.

Equation 7.4. The Standard Equation of an Ellipse: For positive unequal numbers a and b, the equation of an ellipse with center (h,k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Some remarks about Equation 7.4 are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle, Equation 7.1, and divide both sides by r^2, we get

Equation 7.5. The Alternate Standard Equation of a Circle: The equation of a circle with center (h,k) and radius $r > 0$ is

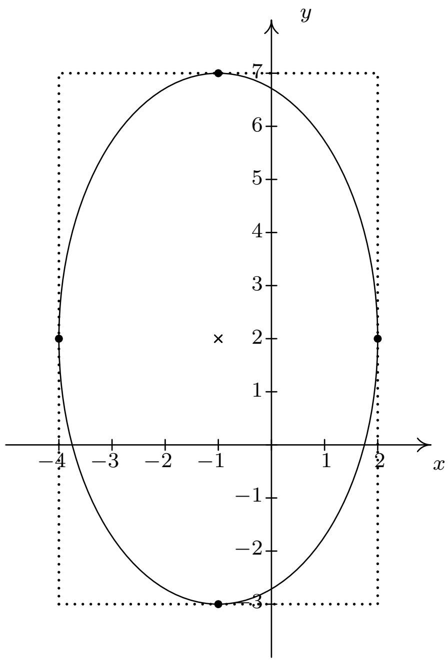
$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

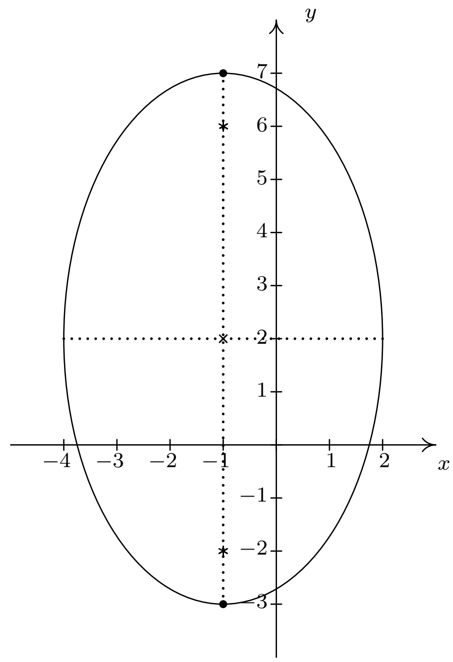
Notice the similarity between Equation 7.4 and Equation 7.5. Both equations involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both Equations 7.4 and 7.5 as shifts and stretches of the Unit Circle $x^2 + y^2 = 1$ in Definition 7.2. Replacing x with $(x-h)$ and y with $(y-k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.[1]

Example 7.4.1. Graph $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 1$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution. We see that this equation is in the standard form of Equation 7.4. Here $x-h$ is $x+1$ so $h = -1$, and $y-k$ is $y-2$ so $k = 2$. Hence, our ellipse is centered at $(-1,2)$. We see that $a^2 = 9$ so $a = 3$, and $b^2 = 25$ so $b = 5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse. As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below on the left.

[1] This was foreshadowed in Exercise 19 in Section 7.2.

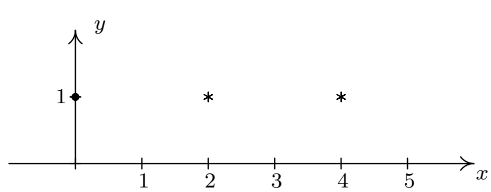




Since we moved farther in the y direction than in the x direction, the major axis will lie along the vertical line $x = -1$, which means the minor axis lies along the horizontal line, $y = 2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, 7)$ and $(-1, -3)$, and the endpoints of the minor axis are $(-4, 2)$ and $(2, 2)$. (Notice these points are the four points we used to draw the guide rectangle.) To find the foci, we find $c = \sqrt{25 - 9} = \sqrt{16} = 4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1, -2)$ and $(-1, 6)$. Plotting all this information gives the graph seen above on the right. $\square$

Example 7.4.2. Find the equation of the ellipse with foci $(2, 1)$ and $(4, 1)$ and vertex $(0, 1)$.

Solution. Plotting the data given to us, we have



From this sketch, we know that the major axis is horizontal, meaning $a > b$. Since the center is the midpoint of the foci, we know it is $(3, 1)$. Since one vertex is $(0, 1)$ we have that $a = 3$, so $a^2 = 9$. All that remains is to find b^2. Since the foci are 1 unit away from the center, we know $c = 1$. Since $a > b$, we have $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$, so $b^2 = 8$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get our final answer to be $\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1$. $\square$

As with circles and parabolas, an equation may be given which is an ellipse, but isn't in the standard form of Equation 7.4. In those cases, as with circles and parabolas before, we will need to massage the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1.

Example 7.4.3. Graph $x^2+4y^2-2x+24y+33=0$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

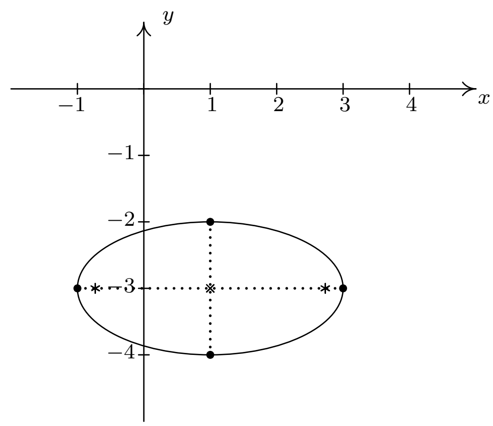
Solution. Since we have a sum of squares and the squared terms have unequal coefficients, it's a good bet we have an ellipse on our hands.[2] We need to complete both squares, and then divide, if necessary, to get the right-hand side equal to 1.

$$\begin{aligned}
x^2+4y^2-2x+24y+33 &= 0 \\
x^2-2x+4y^2+24y &= -33 \\
x^2-2x+4\left(y^2+6y\right) &= -33 \\
\left(x^2-2x+1\right)+4\left(y^2+6y+9\right) &= -33+1+4(9) \\
(x-1)^2+4(y+3)^2 &= 4 \\
\frac{(x-1)^2+4(y+3)^2}{4} &= \frac{4}{4} \\
\frac{(x-1)^2}{4}+(y+3)^2 &= 1 \\
\frac{(x-1)^2}{4}+\frac{(y+3)^2}{1} &= 1
\end{aligned}$$

Now that this equation is in the standard form of Equation 7.4, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+3$ so $k=-3$. Hence, our ellipse is centered at $(1,-3)$. We see that $a^2=4$ so $a=2$, and $b^2=1$ so $b=1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the x direction than in the y direction, the major axis will lie along the horizontal line $y=-3$, which means the minor axis lies along the vertical line $x=1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,-3)$ and $(3,-3)$, and the endpoints of the minor axis are $(1,-2)$ and $(1,-4)$. To find the foci, we find $c=\sqrt{4-1}=\sqrt{3}$, which means

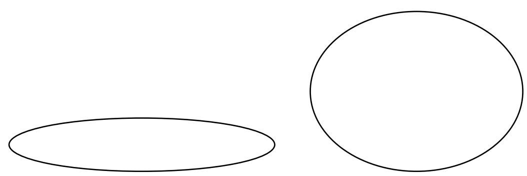
[2]The equation of a parabola has only one squared variable and the equation of a circle has two squared variables with *identical* coefficients.

the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1 - \sqrt{3}, -3)$ and $(1 + \sqrt{3}, -3)$. Plotting all of this information gives



□

As you come across ellipses in the homework exercises and in the wild, you'll notice they come in all shapes in sizes. Compare the two ellipses below.



Certainly, one ellipse is more round than the other. This notion of 'roundness' is quantified below.

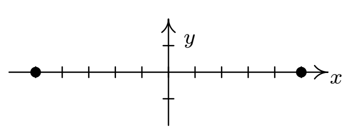
Definition 7.5. The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.98$; for the ellipse above on the right, $e \approx 0.66$. In general, the closer the eccentricity is to 0, the more 'circular' the ellipse; the closer the eccentricity is to 1, the more 'eccentric' the ellipse.

Example 7.4.4. Find the equation of the ellipse whose vertices are $(\pm 5, 0)$ with eccentricity $e = \frac{1}{4}$.

Solution. As before, we plot the data given to us

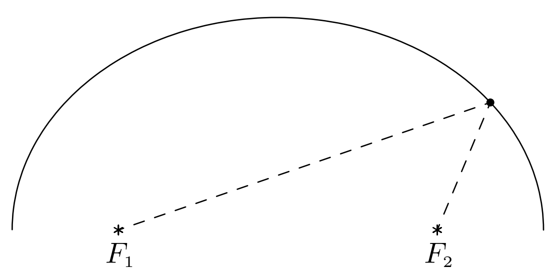


From this sketch, we know that the major axis is horizontal, meaning $a > b$. With the vertices located at $(\pm 5, 0)$, we get $a = 5$ so $a^2 = 25$. We also know that the center is $(0, 0)$ because the center is the midpoint of the vertices. All that remains is to find b^2. To that end, we use the fact that the eccentricity $e = \frac{1}{4}$ which means

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{c}{a} = \frac{c}{5} = \frac{1}{4}$$

from which we get $c = \frac{5}{4}$. To get b^2, we use the fact that $c = \sqrt{a^2 - b^2}$, so $\frac{5}{4} = \sqrt{25 - b^2}$ from which we get $b^2 = \frac{375}{16}$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, yields our final answer $\frac{x^2}{25} + \frac{16y^2}{375} = 1$. □

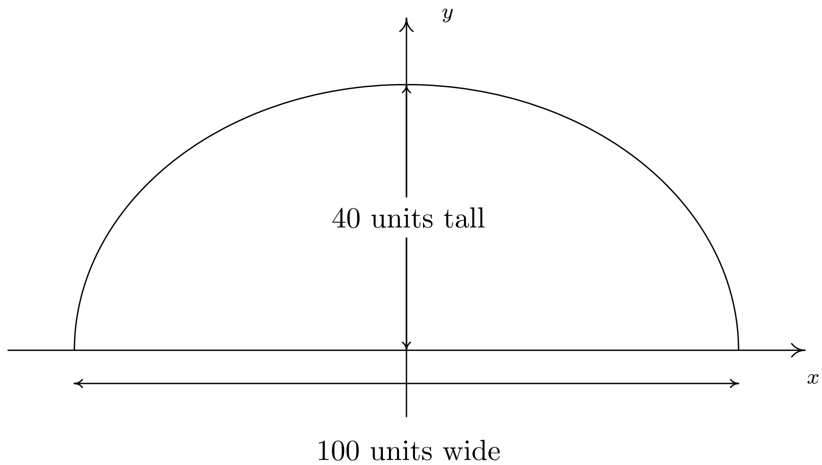
As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus.



Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore the Whispering Galleries in our last example.

Example 7.4.5. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution. Graphing the data yields



It's most convenient to imagine this ellipse centered at $(0,0)$. Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$. Hence, our ellipse has the equation $\frac{x^2}{50^2} + \frac{y^2}{40^2} = 1$. We're looking for the foci, and we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, so that the foci are 30 units from the center. That means they are $50 - 30 = 20$ units from the vertices. Hence, Jason and Jamie should stand 20 feet from opposite ends of the gallery. $\square$

7.4.1 Exercises

In Exercises 1 - 8, graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 14, put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

9. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
10. $12x^2 + 3y^2 - 30y + 39 = 0$
11. $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
12. $x^2 - 2x + 2y^2 - 12y + 3 = 0$
13. $9x^2 + 4y^2 - 4y - 8 = 0$
14. $6x^2 + 5y^2 - 24x + 20y + 14 = 0$

In Exercises 15 - 20, find the standard form of the equation of the ellipse which has the given properties.

15. Center $(3, 7)$, Vertex $(3, 2)$, Focus $(3, 3)$
16. Foci $(0, \pm 5)$, Vertices $(0, \pm 8)$.
17. Foci $(\pm 3, 0)$, length of the Minor Axis 10
18. Vertices $(3, 2)$, $(13, 2)$; Endpoints of the Minor Axis $(8, 4)$, $(8, 0)$
19. Center $(5, 2)$, Vertex $(0, 2)$, eccentricity $\frac{1}{2}$
20. All points on the ellipse are in Quadrant IV except $(0, -9)$ and $(8, 0)$. (One might also say that the ellipse is "tangent to the axes" at those two points.)
21. Repeat Example 7.4.5 for a whispering gallery 200 feet wide and 75 feet tall.
22. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 21 in Section 7.3.

23. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

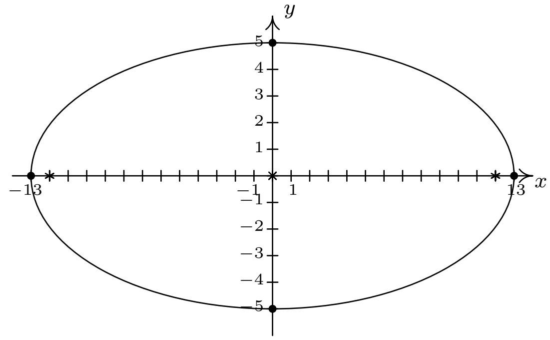
24. The graph of an ellipse clearly fails the Vertical Line Test, Theorem 1.1, so the equation of an ellipse does not define y as a function of x. However, much like with circles and horizontal parabolas, we can split an ellipse into a top half and a bottom half, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the ellipses given in Exercises 1 - 8 above. What difficulties arise when you plot them on the calculator?

25. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, National Statuary Hall in Washington, D.C., and The Cincinnati Museum Center. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in Example 7.4.5?

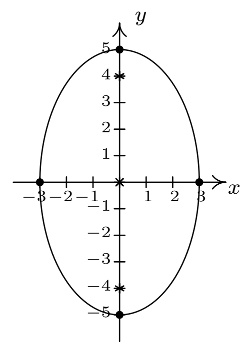
26. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

7.4.2 Answers

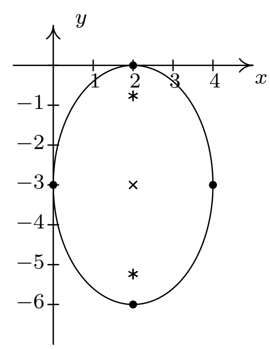
1. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
   Center $(0,0)$
   Major axis along $y = 0$
   Minor axis along $x = 0$
   Vertices $(13,0)$, $(-13,0)$
   Endpoints of Minor Axis $(0,-5)$, $(0,5)$
   Foci $(12,0)$, $(-12,0)$
   $e = \frac{12}{13}$



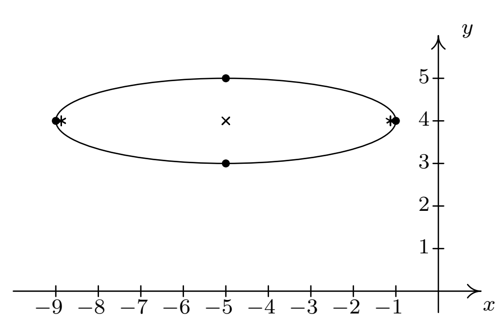
2. $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$
   Center $(0,0)$
   Major axis along $x = 0$
   Minor axis along $y = 0$
   Vertices $(0,5)$, $(0,-5)$
   Endpoints of Minor Axis $(-3,0)$, $(3,0)$
   Foci $(0,-4)$, $(0,4)$
   $e = \frac{4}{5}$



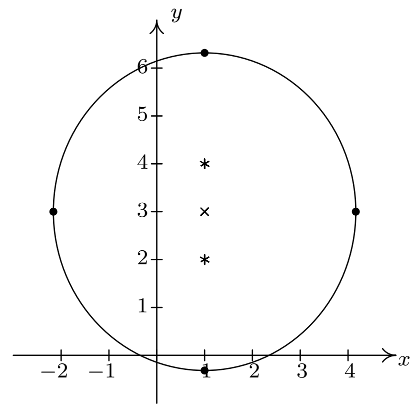
3. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Major axis along $x = 2$
   Minor axis along $y = -3$
   Vertices $(2,0)$, $(2,-6)$
   Endpoints of Minor Axis $(0,-3)$, $(4,-3)$
   Foci $(2,-3+\sqrt{5})$, $(2,-3-\sqrt{5})$
   $e = \frac{\sqrt{5}}{3}$



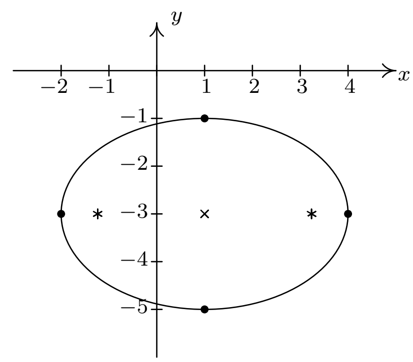
4. $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
   Center $(-5,4)$
   Major axis along $y = 4$
   Minor axis along $x = -5$
   Vertices $(-9,4)$, $(-1,4)$
   Endpoints of Minor Axis $(-5,3)$, $(-5,5)$
   Foci $(-5+\sqrt{15},4)$, $(-5-\sqrt{15},4)$
   $e = \frac{\sqrt{15}}{4}$



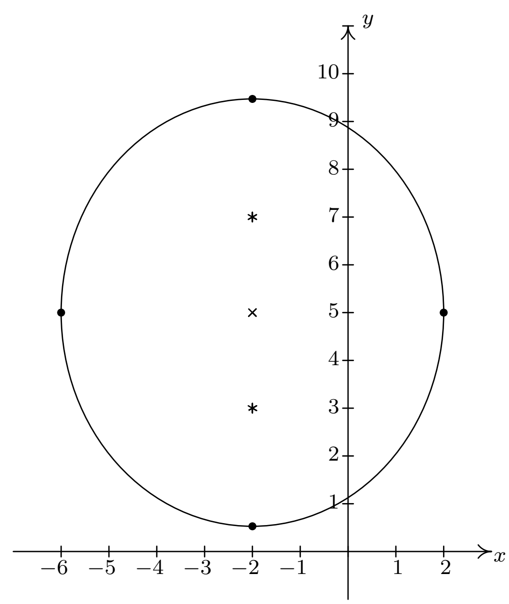
5. $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
Center $(1,3)$
Major axis along $x = 1$
Minor axis along $y = 3$
Vertices $(1, 3+\sqrt{11})$, $(1, 3-\sqrt{11})$
Endpoints of the Minor Axis
$(1-\sqrt{10}, 3)$, $(1+\sqrt{10}, 3)$
Foci $(1,2)$, $(1,4)$
$e = \frac{\sqrt{11}}{11}$



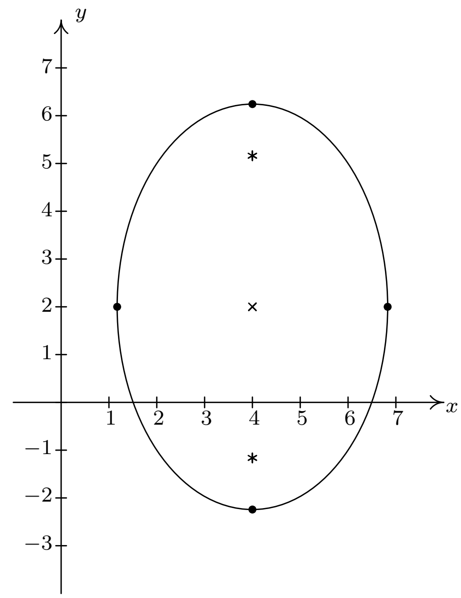
6. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
Center $(1,-3)$
Major axis along $y = -3$
Minor axis along $x = 1$
Vertices $(4,-3)$, $(-2,-3)$
Endpoints of the Minor Axis $(1,-1)$, $(1,-5)$
Foci $(1+\sqrt{5}, -3)$, $(1-\sqrt{5}, -3)$
$e = \frac{\sqrt{5}}{3}$



7. $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
Center $(-2,5)$
Major axis along $x = -2$
Minor axis along $y = 5$
Vertices $(-2, 5+2\sqrt{5})$, $(-2, 5-2\sqrt{5})$
Endpoints of the Minor Axis $(-6,5)$, $(2,5)$
Foci $(-2,7)$, $(-2,3)$
$e = \frac{\sqrt{5}}{5}$



8. $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$
Center $(4, 2)$
Major axis along $x = 4$
Minor axis along $y = 2$
Vertices $(4, 2 + 3\sqrt{2})$, $(4, 2 - 3\sqrt{2})$
Endpoints of the Minor Axis
$(4 - 2\sqrt{2}, 2)$, $(4 + 2\sqrt{2}, 2)$
Foci $(4, 2 + \sqrt{10})$, $(4, 2 - \sqrt{10})$
$e = \frac{\sqrt{5}}{3}$



9. $\dfrac{(x-3)^2}{25} + \dfrac{(y-1)^2}{9} = 1$
Center $(3, 1)$
Major Axis along $y = 1$
Minor Axis along $x = 3$
Vertices $(8, 1)$, $(-2, 1)$
Endpoints of Minor Axis $(3, 4)$, $(3, -2)$
Foci $(7, 1)$, $(-1, 1)$
$e = \frac{4}{5}$

10. $\dfrac{x^2}{3} + \dfrac{(y-5)^2}{12} = 1$
Center $(0, 5)$
Major axis along $x = 0$
Minor axis along $y = 5$
Vertices $(0, 5 - 2\sqrt{3}), (0, 5 + 2\sqrt{3})$
Endpoints of Minor Axis $(-\sqrt{3}, 5)$, $(\sqrt{3}, 5)$
Foci $(0, 2), (0, 8)$
$e = \frac{\sqrt{3}}{2}$

11. $\dfrac{(x-3)^2}{18} + \dfrac{(y+2)^2}{5} = 1$
Center $(3, -2)$
Major axis along $y = -2$
Minor axis along $x = 3$
Vertices $(3 - 3\sqrt{2}, -2), (3 + 3\sqrt{2}, -2)$
Endpoints of Minor Axis $(3, -2 + \sqrt{5})$, $(3, -2 - \sqrt{5})$
Foci $(3 - \sqrt{13}, -2), (3 + \sqrt{13}, -2)$
$e = \frac{\sqrt{26}}{6}$

12. $\dfrac{(x-1)^2}{16} + \dfrac{(y-3)^2}{8} = 1$
Center $(1, 3)$
Major Axis along $y = 3$
Minor Axis along $x = 1$
Vertices $(5, 3)$, $(-3, 3)$
Endpoints of Minor Axis $(1, 3 + 2\sqrt{2})$, $(1, 3 - 2\sqrt{2})$
Foci $(1 + 2\sqrt{2}, 3)$, $(1 - 2\sqrt{2}, 3)$
$e = \frac{\sqrt{2}}{2}$

13. $\dfrac{x^2}{1} + \dfrac{4\left(y-\frac{1}{2}\right)^2}{9} = 1$
Center $\left(0, \frac{1}{2}\right)$
Major Axis along $x = 0$ (the y-axis)
Minor Axis along $y = \frac{1}{2}$
Vertices $(0, 2)$, $(0, -1)$
Endpoints of Minor Axis $\left(-1, \frac{1}{2}\right)$, $\left(1, \frac{1}{2}\right)$
Foci $\left(0, \frac{1+\sqrt{5}}{2}\right)$, $\left(0, \frac{1-\sqrt{5}}{2}\right)$
$e = \frac{\sqrt{5}}{3}$

14. $\dfrac{(x-2)^2}{5} + \dfrac{(y+2)^2}{6} = 1$
Center $(2, -2)$
Major Axis along $x = 2$
Minor Axis along $y = -2$
Vertices $\left(2, -2+\sqrt{6}\right)$, $\left(2, -2-\sqrt{6}\right)$
Endpoints of Minor Axis $\left(2-\sqrt{5}, -2\right)$, $\left(2+\sqrt{5}, -2\right)$
Foci $(2, -1)$, $(2, -3)$
$e = \frac{\sqrt{6}}{6}$

15. $\dfrac{(x-3)^2}{9} + \dfrac{(y-7)^2}{25} = 1$

16. $\dfrac{x^2}{39} + \dfrac{y^2}{64} = 1$

17. $\dfrac{x^2}{34} + \dfrac{y^2}{25} = 1$

18. $\dfrac{(x-8)^2}{25} + \dfrac{(y-2)^2}{4} = 1$

19. $\dfrac{(x-5)^2}{25} + \dfrac{4(y-2)^2}{75} = 1$

20. $\dfrac{(x-8)^2}{64} + \dfrac{(y+9)^2}{81} = 1$

21. Jamie and Jason should stand $100 - 25\sqrt{7} \approx 33.86$ feet from opposite ends of the gallery.

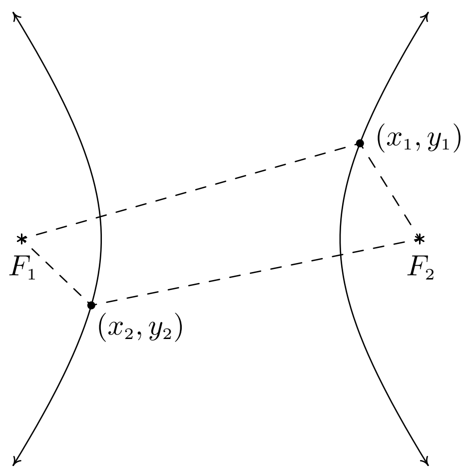
22. The arch can be modeled by the top half of $\frac{x^2}{9} + \frac{y^2}{81} = 1$. One foot in from the base of the arch corresponds to either $x = \pm 2$. Plugging in $x = \pm 2$ gives $y = \pm 3\sqrt{5}$ and since y represents a height, we choose $y = 3\sqrt{5} \approx 6.71$ feet.

23. Distance from the sun to aphelion ≈ 1.0167 AU.
Distance from the sun to perihelion ≈ 0.9833 AU.

7.5 Hyperbolas

In the definition of an ellipse, Definition 7.4, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

Definition 7.6. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



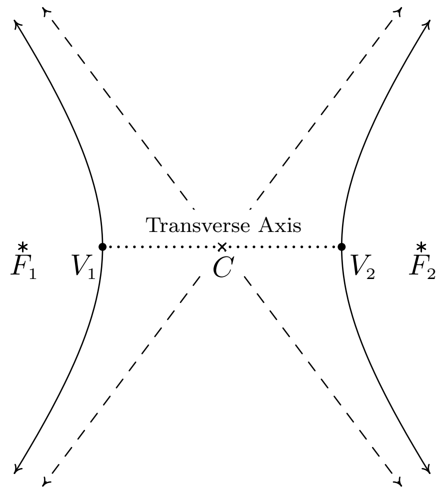
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \quad = \quad d$$

and

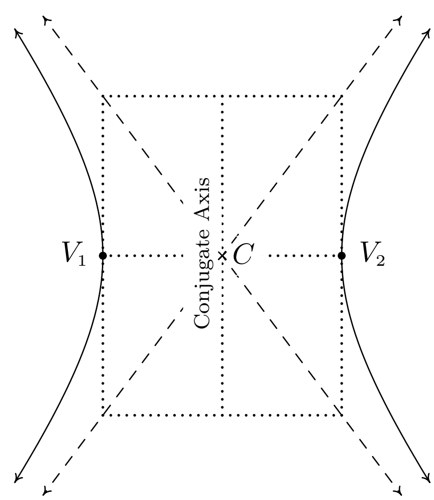
$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \quad = \quad d$$

Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, we will show momentarily that there are lines called **asymptotes** which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



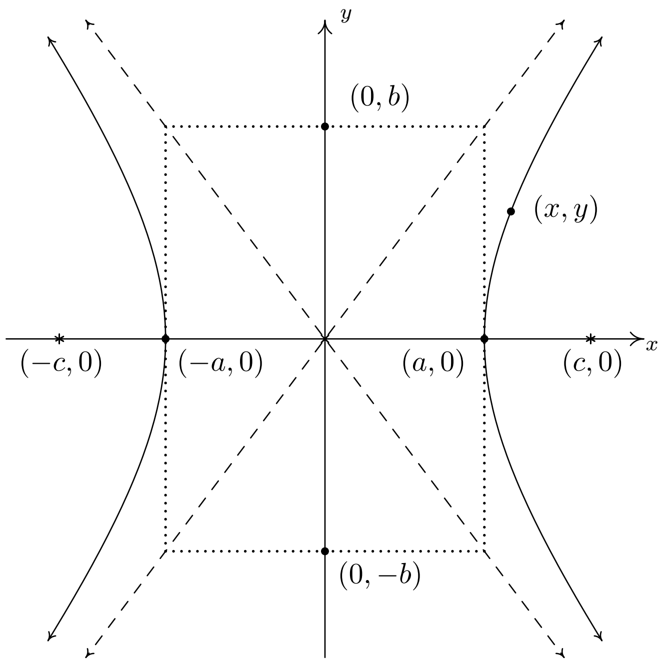
A hyperbola with center C; foci F_1, F_2; and vertices V_1, V_2 and asymptotes (dashed)

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes. In pictures we have



Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle**, much akin to the one we saw Section 7.4 to help us graph ellipses, will aid us in graphing hyperbolas.

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0, 0)$, the vertices are $(a, 0)$ and $(-a, 0)$ and the foci are $(c, 0)$ and $(-c, 0)$. We label the endpoints of the conjugate axis $(0,b)$ and $(0,-b)$. (Although b does not enter into our derivation, we will have to justify this choice as you shall see later.) As before, we assume a, b, and c are all positive numbers. Schematically we have



Since $(a,0)$ is on the hyperbola, it must satisfy the conditions of Definition 7.6. That is, the distance from $(-c,0)$ to $(a,0)$ minus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we get

$$\begin{array}{rcl} \text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0) & = & d \\ (a+c)-(c-a) & = & d \\ 2a & = & d \end{array}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x,y) on the hyperbola. Applying Definition 7.6, we get

$$\begin{array}{rcl} \text{distance from } (-c,0) \text{ to } (x,y) - \text{distance from } (c,0) \text{ to } (x,y) & = & 2a \\ \sqrt{(x-(-c))^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2} & = & 2a \\ \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2} & = & 2a \end{array}$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, Equation 7.4, we arrive at the following.[1]

[1] It is a good exercise to actually work this out.

$$\left(a^2 - c^2\right) x^2 + a^2 y^2 \quad = \quad a^2 \left(a^2 - c^2\right)$$

What remains is to determine the relationship between a, b and c. To that end, we note that since a and c are both positive numbers with $a < c$, we get $a^2 < c^2$ so that $a^2 - c^2$ is a negative number. Hence, $c^2 - a^2$ is a positive number. For reasons which will become clear soon, we re-write the equation by solving for y^2/x^2 to get

$$\begin{aligned}
\left(a^2 - c^2\right) x^2 + a^2 y^2 &= a^2 \left(a^2 - c^2\right) \\
-\left(c^2 - a^2\right) x^2 + a^2 y^2 &= -a^2 \left(c^2 - a^2\right) \\
a^2 y^2 &= \left(c^2 - a^2\right) x^2 - a^2 \left(c^2 - a^2\right) \\
\frac{y^2}{x^2} &= \frac{\left(c^2 - a^2\right)}{a^2} - \frac{\left(c^2 - a^2\right)}{x^2}
\end{aligned}$$

As x and y attain very large values, the quantity $\frac{\left(c^2 - a^2\right)}{x^2} \to 0$ so that $\frac{y^2}{x^2} \to \frac{\left(c^2 - a^2\right)}{a^2}$. By setting $b^2 = c^2 - a^2$ we get $\frac{y^2}{x^2} \to \frac{b^2}{a^2}$. This shows that $y \to \pm \frac{b}{a} x$ as $|x|$ grows large. Thus $y = \pm \frac{b}{a} x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2 - c^2 = -b^2$ which yields

$$\begin{aligned}
\left(a^2 - c^2\right) x^2 + a^2 y^2 &= a^2 \left(a^2 - c^2\right) \\
-b^2 x^2 + a^2 y^2 &= -a^2 b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}$$

The equation above is for a hyperbola whose center is the origin and which opens to the left and right. If the hyperbola were centered at a point (h, k), we would get the following.

Equation 7.6. The Standard Equation of a Horizontal[a] Hyperbola For positive numbers a and b, the equation of a horizontal hyperbola with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

[a]That is, a hyperbola whose branches open to the left and right

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

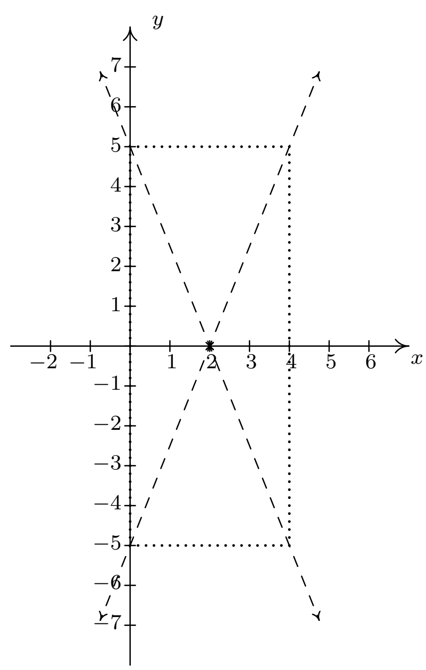
Equation 7.7. The Standard Equation of a Vertical Hyperbola For positive numbers a and b, the equation of a vertical hyperbola with center (h, k) is:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

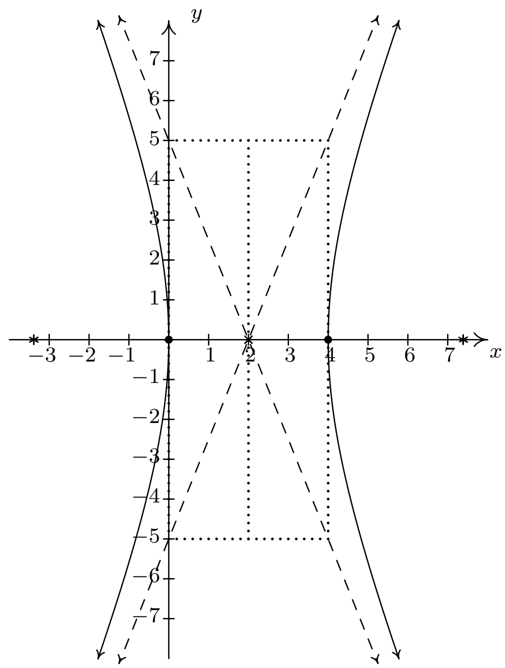
The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, as seen in the derivation, can be found by the formula $c = \sqrt{a^2+b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a circle or ellipse because the hyperbola formula involves a **difference** of squares where the circle and ellipse formulas both involve the **sum** of squares.

Example 7.5.1. Graph the equation $\frac{(x-2)^2}{4} - \frac{y^2}{25} = 1$. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

Solution. We first see that this equation is given to us in the standard form of Equation 7.6. Here $x-h$ is $x-2$ so $h=2$, and $y-k$ is y so $k=0$. Hence, our hyperbola is centered at $(2,0)$. We see that $a^2 = 4$ so $a=2$, and $b^2 = 25$ so $b=5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle. This yields the following set up.



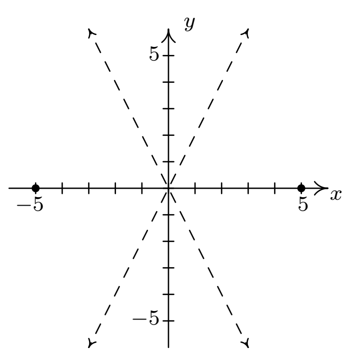
Since the y^2 term is being subtracted from the x^2 term, we know that the branches of the hyperbola open to the left and right. This means that the transverse axis lies along the x-axis. Hence, the conjugate axis lies along the vertical line $x=2$. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are 2 units to the left and right of $(2,0)$ at $(0,0)$ and $(4,0)$. To find the foci, we need $c = \sqrt{a^2+b^2} = \sqrt{4+25} = \sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2,0)$ to arrive at $(2-\sqrt{29},0)$ (approximately $(-3.39,0)$) and $(2+\sqrt{29},0)$ (approximately $(7.39,0)$). To determine the equations of the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(2,0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{5}{2}$. Using the point-slope equation of a line, Equation 2.2, yields $y - 0 = \pm\frac{5}{2}(x-2)$, so we get $y = \frac{5}{2}x - 5$ and $y = -\frac{5}{2}x + 5$. Putting it all together, we get



□

Example 7.5.2. Find the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(\pm 5, 0)$.

Solution. Plotting the data given to us, we have



This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0,0)$. Hence, our standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since the vertices are $(\pm 5, 0)$, we have $a = 5$ so $a^2 = 25$. In order to determine b^2, we recall that the slopes of the asymptotes are $\pm\frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$ and our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$. □

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side
2. Complete the square in both variables as needed
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1

Example 7.5.3. Consider the equation $9y^2 - x^2 - 6x = 10$. Put this equation in to standard form and graph. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci, and the equations of the asymptotes.

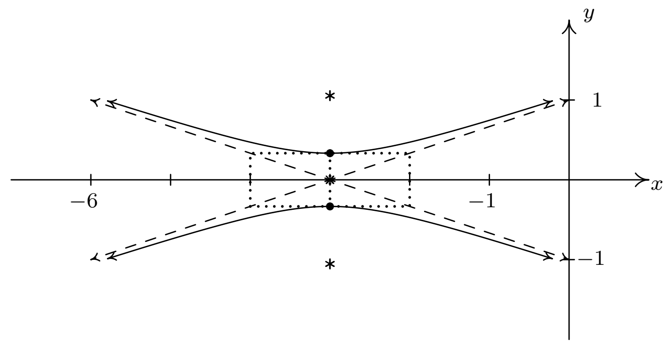
Solution. We need only complete the square on x:

$$\begin{aligned} 9y^2 - x^2 - 6x &= 10 \\ 9y^2 - 1\left(x^2 + 6x\right) &= 10 \\ 9y^2 - \left(x^2 + 6x + 9\right) &= 10 - 1(9) \\ 9y^2 - (x+3)^2 &= 1 \\ \frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.7, we see that $x - h$ is $x + 3$ so $h = -3$, and $y - k$ is y so $k = 0$. Hence, our hyperbola is centered at $(-3, 0)$. We find that $a^2 = 1$ so $a = 1$, and $b^2 = \frac{1}{9}$ so $b = \frac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle. Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x = -3$ and the conjugate axis lies along the x-axis. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $\frac{1}{3}$ of a unit above and below $(-3, 0)$ at $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$. To find the foci, we use

$$c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$$

Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3, 0)$ to arrive at $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$. To determine the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(-3, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Using the point-slope equation of a line, Equation 2.2, we get $y = \frac{1}{3}x + 1$ and $y = -\frac{1}{3}x - 1$. Putting it all together, we get



□

Hyperbolas can be used in so-called 'trilateration,' or 'positioning' problems. The procedure outlined in the next example is the basis of the (now virtually defunct) LOng Range Aid to Navigation (LORAN for short) system.[2]

Example 7.5.4. Jeff is stationed 10 miles due west of Carl in an otherwise empty forest in an attempt to locate an elusive Sasquatch. At the stroke of midnight, Jeff records a Sasquatch call 9 seconds earlier than Carl. If the speed of sound that night is 760 miles per hour, determine a hyperbolic path along which Sasquatch must be located.

Solution. Since Jeff hears Sasquatch sooner, it is closer to Jeff than it is to Carl. Since the speed of sound is 760 miles per hour, we can determine how much closer Sasquatch is to Jeff by multiplying

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} \times 9\,\text{seconds} = 1.9\,\text{miles}$$

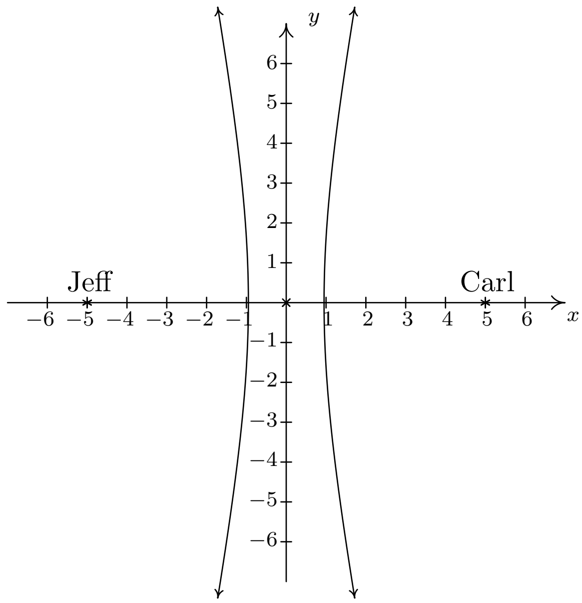
This means that Sasquatch is 1.9 miles closer to Jeff than it is to Carl. In other words, Sasquatch must lie on a path where

$$(\text{the distance to Carl}) - (\text{the distance to Jeff}) = 1.9$$

This is exactly the situation in the definition of a hyperbola, Definition 7.6. In this case, Jeff and Carl are located at the foci,[3] and our fixed distance d is 1.9. For simplicity, we assume the hyperbola is centered at $(0,0)$ with its foci at $(-5,0)$ and $(5,0)$. Schematically, we have

[2]GPS now rules the positioning kingdom. Is there still a place for LORAN and other land-based systems? Do satellites ever malfunction?

[3]We usually like to be the *center* of attention, but being the *focus* of attention works equally well.



We are seeking a curve of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which the distance from the center to each focus is $c = 5$. As we saw in the derivation of the standard equation of the hyperbola, Equation 7.6, $d = 2a$, so that $2a = 1.9$, or $a = 0.95$ and $a^2 = 0.9025$. All that remains is to find b^2. To that end, we recall that $a^2 + b^2 = c^2$ so $b^2 = c^2 - a^2 = 25 - 0.9025 = 24.0975$. Since Sasquatch is closer to Jeff than it is to Carl, it must be on the western (left hand) branch of $\frac{x^2}{0.9025} - \frac{y^2}{24.0975} = 1$. $\square$

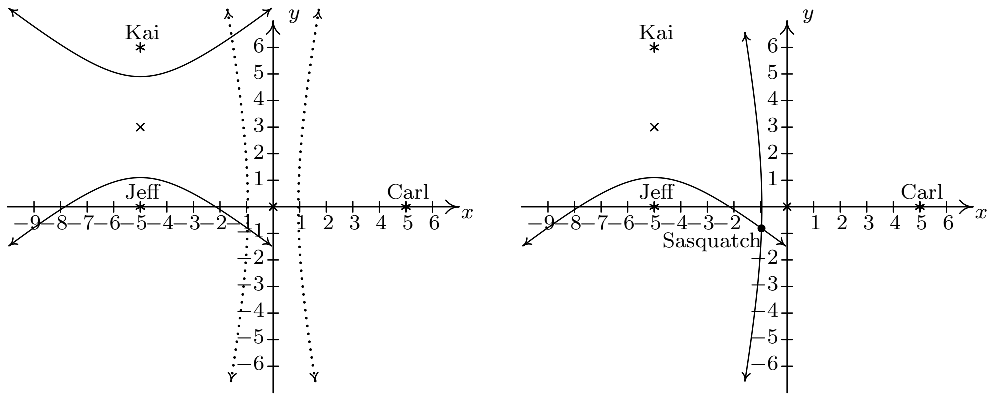
In our previous example, we did not have enough information to pin down the exact location of Sasquatch. To accomplish this, we would need a third observer.

Example 7.5.5. By a stroke of luck, Kai was also camping in the woods during the events of the previous example. He was located 6 miles due north of Jeff and heard the Sasquatch call 18 seconds after Jeff did. Use this added information to locate Sasquatch.

Solution. Kai and Jeff are now the foci of a second hyperbola where the fixed distance d can be determined as before

$$760\,\frac{\text{miles}}{\text{hour}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} \times 18\,\text{seconds} = 3.8\,\text{miles}$$

Since Jeff was positioned at $(-5, 0)$, we place Kai at $(-5, 6)$. This puts the center of the new hyperbola at $(-5, 3)$. Plotting Kai's position and the new center gives us the diagram below on the left. The second hyperbola is vertical, so it must be of the form $\frac{(y-3)^2}{b^2} - \frac{(x+5)^2}{a^2} = 1$. As before, the distance d is the length of the major axis, which in this case is $2b$. We get $2b = 3.8$ so that $b = 1.9$ and $b^2 = 3.61$. With Kai 6 miles due North of Jeff, we have that the distance from the center to the focus is $c = 3$. Since $a^2 + b^2 = c^2$, we get $a^2 = c^2 - b^2 = 9 - 3.61 = 5.39$. Kai heard the Sasquatch call after Jeff, so Kai is farther from Sasquatch than Jeff. Thus Sasquatch must lie on the southern branch of the hyperbola $\frac{(y-3)^2}{3.61} - \frac{(x+5)^2}{5.39} = 1$. Looking at the western branch of the hyperbola determined by Jeff and Carl along with the southern branch of the hyperbola determined by Kai and Jeff, we see that there is exactly one point in common, and this is where Sasquatch must have been when it called.



To determine the coordinates of this point of intersection exactly, we would need techniques for solving systems of non-linear equations (which we won't see until Section 8.7), so we use the calculator[4] Doing so, we get Sasquatch is approximately at $(-0.9629, -0.8113)$. □

Each of the conic sections we have studied in this chapter result from graphing equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ for different choices of A, C, D, E, and[5] F. While we've seen examples[6] demonstrate *how* to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about *which* standard form to choose.[7]

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a non-degenerate conic section.[a]

- If just *one* variable is squared, the graph is a parabola. Put the equation in the form of Equation 7.2 (if x is squared) or Equation 7.3 (if y is squared).

If *both* variables are squared, look at the coefficients of x^2 and y^2, A and B.

- If $A = B$, the graph is a circle. Put the equation in the form of Equation 7.1.
- If $A \neq B$ but A and B have the *same sign*, the graph is an ellipse. Put the equation in the form of Equation 7.4.
- If A and B have the *different signs*, the graph is a hyperbola. Put the equation in the form of either Equation 7.6 or Equation 7.7.

[a]That is, a parabola, circle, ellipse, or hyperbola – see Section 7.1.

[4]First solve each hyperbola for y, and choose the correct equation (branch) before proceeding.
[5]See Section 11.6 to see why we skip B.
[6]Examples 7.2.3, 7.3.4, 7.4.3, and 7.5.3, in particular.
[7]We formalize this in Exercise 34.

7.5.1 Exercises

In Exercises 1 - 8, graph the hyperbola. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$

In Exercises 9 - 12, put the equation in standard form. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

9. $12x^2 - 3y^2 + 30y - 111 = 0$
10. $18y^2 - 5x^2 + 72y + 30x - 63 = 0$
11. $9x^2 - 25y^2 - 54x - 50y - 169 = 0$
12. $-6x^2 + 5y^2 - 24x + 40y + 26 = 0$

In Exercises 13 - 18, find the standard form of the equation of the hyperbola which has the given properties.

13. Center $(3,7)$, Vertex $(3,3)$, Focus $(3,2)$
14. Vertex $(0,1)$, Vertex $(8,1)$, Focus $(-3,1)$
15. Foci $(0,\pm 8)$, Vertices $(0,\pm 5)$.
16. Foci $(\pm 5,0)$, length of the Conjugate Axis 6
17. Vertices $(3,2)$, $(13,2)$; Endpoints of the Conjugate Axis $(8,4)$, $(8,0)$
18. Vertex $(-10,5)$, Asymptotes $y = \pm\frac{1}{2}(x-6)+5$

In Exercises 19 - 28, find the standard form of the equation using the guidelines on page 540 and then graph the conic section.

19. $x^2 - 2x - 4y - 11 = 0$
20. $x^2 + y^2 - 8x + 4y + 11 = 0$
21. $9x^2 + 4y^2 - 36x + 24y + 36 = 0$
22. $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

23. $y^2 + 8y - 4x + 16 = 0$

24. $4x^2 + y^2 - 8x + 4 = 0$

25. $4x^2 + 9y^2 - 8x + 54y + 49 = 0$

26. $x^2 + y^2 - 6x + 4y + 14 = 0$

27. $2x^2 + 4y^2 + 12x - 8y + 25 = 0$

28. $4x^2 - 5y^2 - 40x - 20y + 160 = 0$

29. The graph of a vertical or horizontal hyperbola clearly fails the Vertical Line Test, Theorem 1.1, so the equation of a vertical of horizontal hyperbola does not define y as a function of x.[8] However, much like with circles, horizontal parabolas and ellipses, we can split a hyperbola into pieces, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the hyperbolas given in Exercises 1 - 8 above. How many pieces do you need for a vertical hyperbola? How many for a horizontal hyperbola?

30. The location of an earthquake's epicenter − the point on the surface of the Earth directly above where the earthquake actually occurred − can be determined by a process similar to how we located Sasquatch in Example 7.5.5. (As we said back in Exercise 75 in Section 6.1, earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a course in Geology or the U.S. Geological Survey's Earthquake Hazards Program found here.) Our technique works only for relatively small distances because we need to assume that the Earth is flat in order to use hyperbolas in the plane.[9] The P-waves ("P" stands for Primary) of an earthquake in Sasquatchia travel at 6 kilometers per second.[10] Station A records the waves first. Then Station B, which is 100 kilometers due north of Station A, records the waves 2 seconds later. Station C, which is 150 kilometers due west of Station A records the waves 3 seconds after that (a total of 5 seconds after Station A). Where is the epicenter?

31. The notion of eccentricity introduced for ellipses in Definition 7.5 in Section 7.4 is the same for hyperbolas in that we can define the eccentricity e of a hyperbola as

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

    (a) With the help of your classmates, explain why $e > 1$ for any hyperbola.
    (b) Find the equation of the hyperbola with vertices $(\pm 3, 0)$ and eccentricity $e = 2$.
    (c) With the help of your classmates, find the eccentricity of each of the hyperbolas in Exercises 1 - 8. What role does eccentricity play in the shape of the graphs?

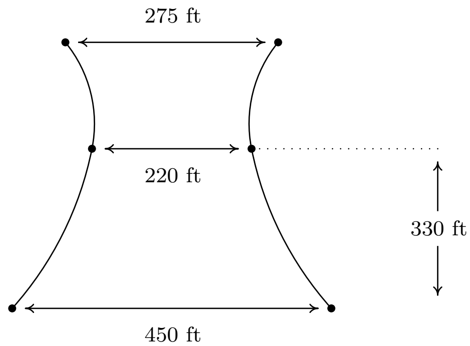
32. On page 510 in Section 7.3, we discussed paraboloids of revolution when studying the design of satellite dishes and parabolic mirrors. In much the same way, 'natural draft' cooling towers are often shaped as **hyperboloids of revolution**. Each vertical cross section of these towers

[8] We will see later in the text that the graphs of certain rotated hyperbolas pass the Vertical Line Test.

[9] Back in the Exercises in Section 1.1 you were asked to research people who believe the world is flat. What did you discover?

[10] Depending on the composition of the crust at a specific location, P-waves can travel between 5 kps and 8 kps.

is a hyperbola. Suppose the a natural draft cooling tower has the cross section below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.



33. With the help of your classmates, research the Cassegrain Telescope. It uses the reflective property of the hyperbola as well as that of the parabola to make an ingenious telescope.

34. With the help of your classmates show that if $Ax^2 + Cy^2 + Dx + Ey + F = 0$ determines a non-degenerate conic[11] then

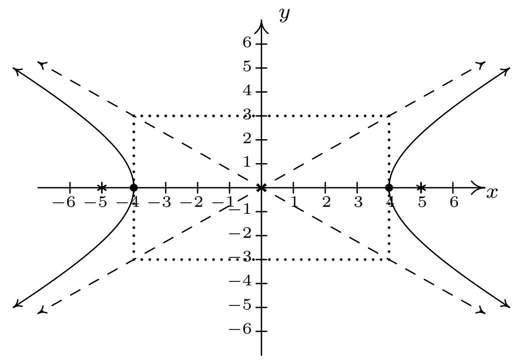
    - $AC < 0$ means that the graph is a hyperbola
    - $AC = 0$ means that the graph is a parabola
    - $AC > 0$ means that the graph is an ellipse or circle

    NOTE: This result will be generalized in Theorem 11.11 in Section 11.6.1.

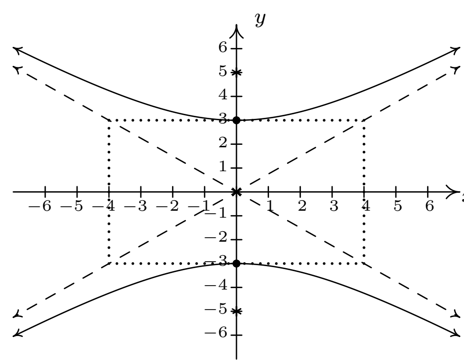
[11] Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

7.5.2 Answers

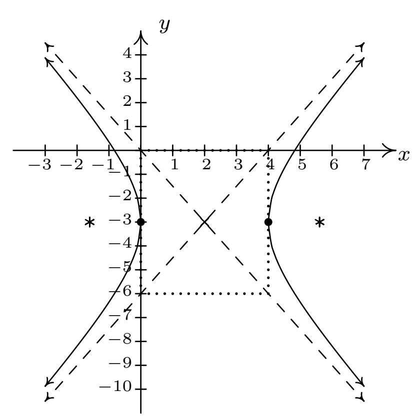
1. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$
   Center $(0,0)$
   Transverse axis on $y=0$
   Conjugate axis on $x=0$
   Vertices $(4,0),(-4,0)$
   Foci $(5,0),(-5,0)$
   Asymptotes $y = \pm\frac{3}{4}x$



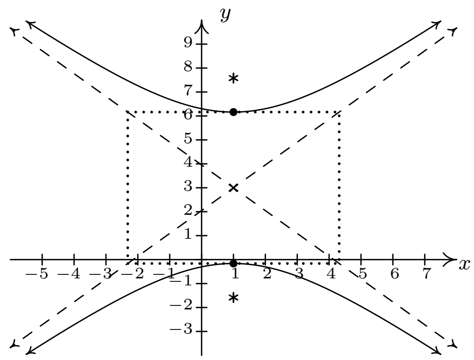
2. $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$
   Center $(0,0)$
   Transverse axis on $x=0$
   Conjugate axis on $y=0$
   Vertices $(0,3),(0,-3)$
   Foci $(0,5),(0,-5)$
   Asymptotes $y = \pm\frac{3}{4}x$



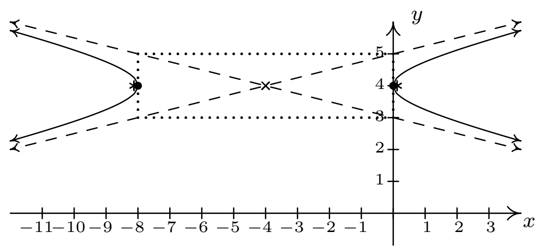
3. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$
   Center $(2,-3)$
   Transverse axis on $y=-3$
   Conjugate axis on $x=2$
   Vertices $(0,-3),(4,-3)$
   Foci $(2+\sqrt{13},-3),(2-\sqrt{13},-3)$
   Asymptotes $y = \pm\frac{3}{2}(x-2)-3$



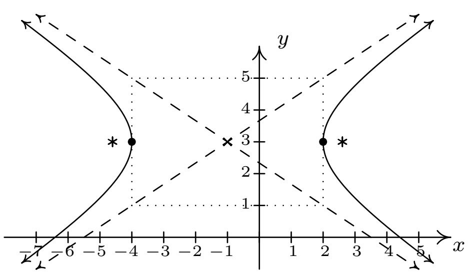
4. $\dfrac{(y-3)^2}{11} - \dfrac{(x-1)^2}{10} = 1$
Center $(1,3)$
Transverse axis on $x=1$
Conjugate axis on $y=3$
Vertices $(1, 3+\sqrt{11}), (1, 3-\sqrt{11})$
Foci $(1, 3+\sqrt{21}), (1, 3-\sqrt{21})$
Asymptotes $y = \pm\frac{\sqrt{110}}{10}(x-1)+3$



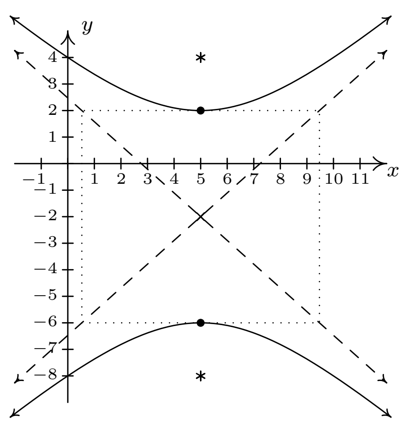
5. $\dfrac{(x+4)^2}{16} - \dfrac{(y-4)^2}{1} = 1$
Center $(-4,4)$
Transverse axis on $y=4$
Conjugate axis on $x=-4$
Vertices $(-8,4), (0,4)$
Foci $(-4+\sqrt{17}, 4), (-4-\sqrt{17}, 4)$
Asymptotes $y = \pm\frac{1}{4}(x+4)+4$



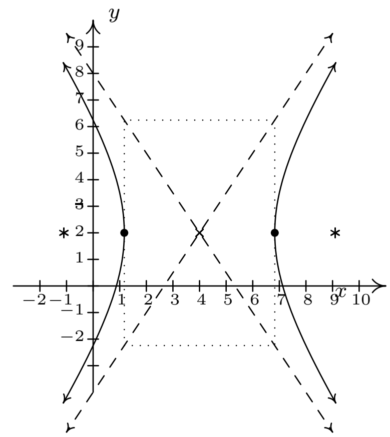
6. $\dfrac{(x+1)^2}{9} - \dfrac{(y-3)^2}{4} = 1$
Center $(-1,3)$
Transverse axis on $y=3$
Conjugate axis on $x=-1$
Vertices $(2,3), (-4,3)$
Foci $\left(-1+\sqrt{13}, 3\right), \left(-1-\sqrt{13}, 3\right)$
Asymptotes $y = \pm\frac{2}{3}(x+1)+3$



7. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$
Center $(5,-2)$
Transverse axis on $x=5$
Conjugate axis on $y=-2$
Vertices $(5,2), (5,-6)$
Foci $(5,4), (5,-8)$
Asymptotes $y = \pm\frac{2\sqrt{5}}{5}(x-5)-2$



8. $\dfrac{(x-4)^2}{8} - \dfrac{(y-2)^2}{18} = 1$
Center $(4,2)$
Transverse axis on $y=2$
Conjugate axis on $x=4$
Vertices $\left(4+2\sqrt{2},2\right),\left(4-2\sqrt{2},2\right)$
Foci $\left(4+\sqrt{26},2\right),\left(4-\sqrt{26},2\right)$
Asymptotes $y=\pm\frac{3}{2}(x-4)+2$



9. $\dfrac{x^2}{3} - \dfrac{(y-5)^2}{12} = 1$
Center $(0,5)$
Transverse axis on $y=5$
Conjugate axis on $x=0$
Vertices $(\sqrt{3},5),(-\sqrt{3},5)$
Foci $(\sqrt{15},5),(-\sqrt{15},5)$
Asymptotes $y=\pm 2x+5$

10. $\dfrac{(y+2)^2}{5} - \dfrac{(x-3)^2}{18} = 1$
Center $(3,-2)$
Transverse axis on $x=3$
Conjugate axis on $y=-2$
Vertices $(3,-2+\sqrt{5}),(3,-2-\sqrt{5})$
Foci $(3,-2+\sqrt{23}),(3,-2-\sqrt{23})$
Asymptotes $y=\pm\frac{\sqrt{10}}{6}(x-3)-2$

11. $\dfrac{(x-3)^2}{25} - \dfrac{(y+1)^2}{9} = 1$
Center $(3,-1)$
Transverse axis on $y=-1$
Conjugate axis on $x=3$
Vertices $(8,-1),(-2,-1)$
Foci $\left(3+\sqrt{34},-1\right),\left(3-\sqrt{34},-1\right)$
Asymptotes $y=\pm\frac{3}{5}(x-3)-1$

12. $\dfrac{(y+4)^2}{6} - \dfrac{(x+2)^2}{5} = 1$
Center $(-2,-4)$
Transverse axis on $x=-2$
Conjugate axis on $y=-4$
Vertices $\left(-2,-4+\sqrt{6}\right),\left(-2,-4-\sqrt{6}\right)$
Foci $\left(-2,-4+\sqrt{11}\right),\left(-2,-4-\sqrt{11}\right)$
Asymptotes $y=\pm\frac{\sqrt{30}}{5}(x+2)-4$

13. $\dfrac{(y-7)^2}{16} - \dfrac{(x-3)^2}{9} = 1$

14. $\dfrac{(x-4)^2}{16} - \dfrac{(y-1)^2}{33} = 1$

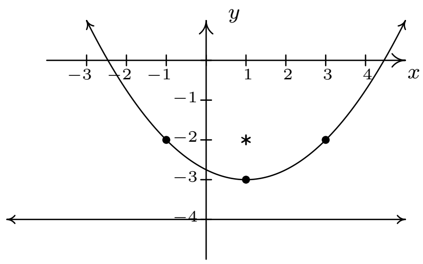
15. $\dfrac{y^2}{25} - \dfrac{x^2}{39} = 1$

16. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

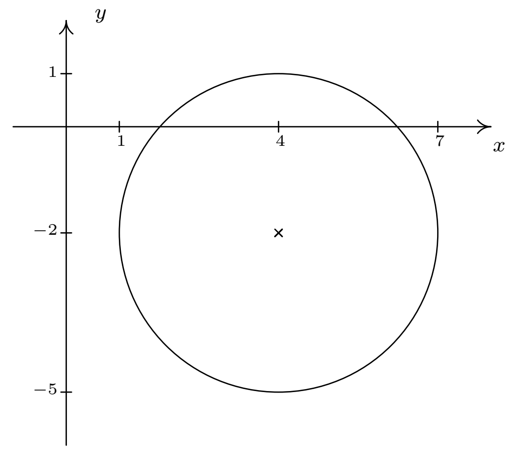
17. $\dfrac{(x-8)^2}{25} - \dfrac{(y-2)^2}{4} = 1$

18. $\dfrac{(x-6)^2}{256} - \dfrac{(y-5)^2}{64} = 1$

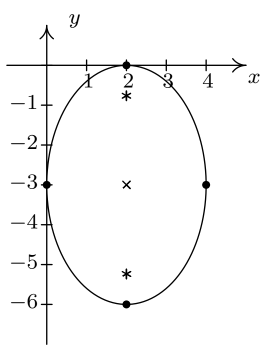
19. $(x-1)^2 = 4(y+3)$



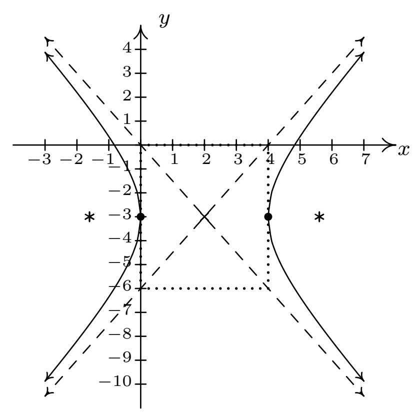
20. $(x-4)^2 + (y+2)^2 = 9$



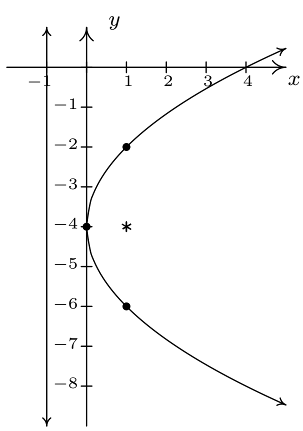
21. $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$



22. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$



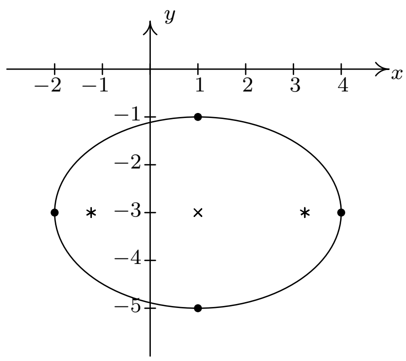
23. $(y+4)^2 = 4x$



24. $\dfrac{(x-1)^2}{1} + \dfrac{y^2}{4} = 0$

The graph is the point $(1,0)$ only.

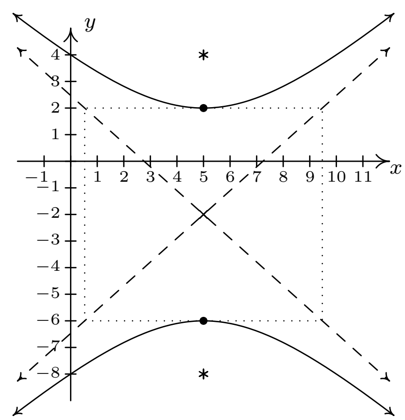
25. $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$



26. $(x-3)^2 + (y+2)^2 = -1$
There is no graph.

27. $\dfrac{(x+3)^2}{2} + \dfrac{(y-1)^2}{1} = -\dfrac{3}{4}$
There is no graph.

28. $\dfrac{(y+2)^2}{16} - \dfrac{(x-5)^2}{20} = 1$



30. By placing Station A at $(0, -50)$ and Station B at $(0, 50)$, the two second time difference yields the hyperbola $\frac{y^2}{36} - \frac{x^2}{2464} = 1$ with foci A and B and center $(0, 0)$. Placing Station C at $(-150, -50)$ and using foci A and C gives us a center of $(-75, -50)$ and the hyperbola $\frac{(x+75)^2}{225} - \frac{(y+50)^2}{5400} = 1$. The point of intersection of these two hyperbolas which is closer to A than B and closer to A than C is $(-57.8444, -9.21336)$ so that is the epicenter.

31. (b) $\dfrac{x^2}{9} - \dfrac{y^2}{27} = 1$.

32. The tower may be modeled (approximately)[12] by $\frac{x^2}{12100} - \frac{(y-330)^2}{34203} = 1$. To find the height, we plug in $x = 137.5$ which yields $y \approx 191$ or $y \approx 469$. Since the top of the tower is above the narrowest point, we get the tower is approximately 469 feet tall.

[12] The exact value underneath $(y-330)^2$ is $\frac{52707600}{1541}$ in case you need more precision.

Chapter 8

Systems of Equations and Matrices

8.1 Systems of Linear Equations: Gaussian Elimination

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The x-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the y-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to x and y are of interest, we have what is known as a **system of equations**, usually written as

$$\left\{ \begin{array}{rcl} y & = & f(x) \\ y & = & g(x) \end{array} \right.$$

The 'curly bracket' notation means we are to find all **pairs** of points (x, y) which satisfy **both** equations. We begin our study of systems of equations by reviewing some basic notions from Intermediate Algebra.

Definition 8.1. A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where a_1, a_2 and c are real numbers and at least one of a_1 and a_2 is nonzero.

For reasons which will become clear later in the section, we are using subscripts in Definition 8.1 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables[1] by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$. If a_1 and a_2 are both 0, then depending on c, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play

[1] Critics may argue that $x = 5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

in the upcoming sections, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2+y=1$, $xy=5$ and $e^{2x}+\ln(y)=1$. We leave it to the reader to explain why these do not satisfy Definition 8.1. From what we know from Sections 1.2 and 2.1, the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example reviews some of the basic techniques first learned in Intermediate Algebra.

Example 8.1.1. Solve the following systems of equations. Check your answer algebraically and graphically.

1. $\left\{ \begin{array}{rcr} 2x - y & = & 1 \\ y & = & 3 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} 3x + 4y & = & -2 \\ -3x - y & = & 5 \end{array} \right.$
3. $\left\{ \begin{array}{rcr} \frac{x}{3} - \frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} & = & \frac{1}{2} \end{array} \right.$
4. $\left\{ \begin{array}{rcr} 2x - 4y & = & 6 \\ 3x - 6y & = & 9 \end{array} \right.$
5. $\left\{ \begin{array}{rcr} 6x + 3y & = & 9 \\ 4x + 2y & = & 12 \end{array} \right.$
6. $\left\{ \begin{array}{rcr} x - y & = & 0 \\ x + y & = & 2 \\ -2x + y & = & -2 \end{array} \right.$

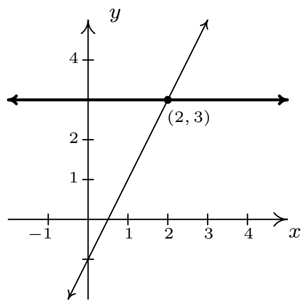
Solution.

1. Our first system is nearly solved for us. The second equation tells us that $y=3$. To find the corresponding value of x, we **substitute** this value for y into the the first equation to obtain $2x-3=1$, so that $x=2$. Our solution to the system is $(2,3)$. To check this algebraically, we substitute $x=2$ and $y=3$ into each equation and see that they are satisfied. We see $2(2)-3=1$, and $3=3$, as required. To check our answer graphically, we graph the lines $2x-y=1$ and $y=3$ and verify that they intersect at $(2,3)$.

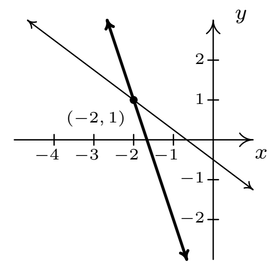
2. To solve the second system, we use the **addition** method to **eliminate** the variable x. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rrcr} & 3x + 4y & = & -2 \\ + & (-3x - y & = & 5) \\ \hline & 3y & = & 3 \end{array}$$

This gives us $y=1$. We now substitute $y=1$ into either of the two equations, say $-3x-y=5$, to get $-3x-1=5$ so that $x=-2$. Our solution is $(-2,1)$. Substituting $x=-2$ and $y=1$ into the first equation gives $3(-2)+4(1)=-2$, which is true, and, likewise, when we check $(-2,1)$ in the second equation, we get $-3(-2)-1=5$, which is also true. Geometrically, the lines $3x+4y=-2$ and $-3x-y=5$ intersect at $(-2,1)$.



$2x - y = 1$
$\boldsymbol{y = 3}$



$3x + 4y = -2$
$\boldsymbol{-3x - y = 5}$

3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18 to obtain the kinder, gentler system

$$\left\{ \begin{array}{rcr} 5x - 12y & = & 21 \\ 4x + 6y & = & 9 \end{array} \right.$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by -5, we will be in a position to eliminate the x term

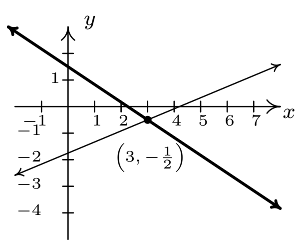
$$\begin{array}{rrcr} & 20x - 48y & = & 84 \\ + & (-20x - 30y & = & -45) \\ \hline & -78y & = & 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$. At this point, we have no choice $-$ in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$.

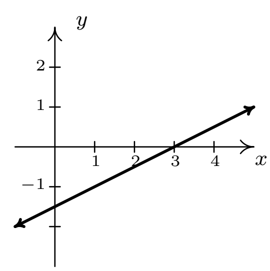
4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and both sides of the second equation by -2, we are ready to eliminate the x

$$\begin{array}{rrcr} & 6x-12y & = & 18 \\ + & (-6x+12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

We eliminated not only the x, but the y as well and we are left with the identity $0=0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair (x,y) satisfies the equation $2x-4y=6$, it *automatically* satisfies the equation $3x-6y=9$. One way to describe the solution set to this system is to use the roster method[2] and write $\{(x,y)\,|\,2x-4y=6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution to a system**. Our first step is to solve $2x-4y=6$ for one of the variables, say $y=\frac{1}{2}x-\frac{3}{2}$. For each value of x, the formula $y=\frac{1}{2}x-\frac{3}{2}$ determines the corresponding y-value of a solution. Since we have no restriction on x, it is called a **free variable**. We let $x=t$, a so-called 'parameter', and get $y=\frac{1}{2}t-\frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t,\frac{1}{2}t-\frac{3}{2}\right)\mid -\infty<t<\infty\right\}$.[3] For specific values of t, we can generate solutions. For example, $t=0$ gives us the solution $\left(0,-\frac{3}{2}\right)$; $t=117$ gives us $(117,57)$, and while we can readily check each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always. We claim that for any real number t, the pair $\left(t,\frac{1}{2}t-\frac{3}{2}\right)$ satisfies both equations. Substituting $x=t$ and $y=\frac{1}{2}t-\frac{3}{2}$ into $2x-4y=6$ gives $2t-4\left(\frac{1}{2}t-\frac{3}{2}\right)=6$. Simplifying, we get $2t-2t+6=6$, which is always true. Similarly, when we make these substitutions in the equation $3x-6y=9$, we get $3t-6\left(\frac{1}{2}t-\frac{3}{2}\right)=9$ which reduces to $3t-3t+9=9$, so it checks out, too. Geometrically, $2x-4y=6$ and $3x-6y=9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.



$\frac{x}{3}-\frac{4y}{5}=\frac{7}{5}$
$\boldsymbol{\frac{2x}{9}+\frac{y}{3}=\frac{1}{2}}$



$2x-4y=6$
$\mathbf{3x-6y=9}$
(Same line.)

[2] See Section 1.2 for a review of this.

[3] Note that we could have just as easily chosen to solve $2x-4y=6$ for x to obtain $x=2y+3$. Letting y be the parameter t, we have that for any value of t, $x=2t+3$, which gives $\{(2t+3,t)\,|\,-\infty<t<\infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by -3, we set the stage to eliminate x

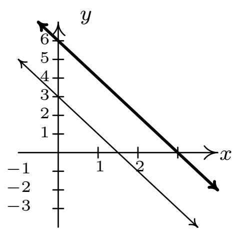
$$\begin{array}{rrcr} & 12x+6y & = & 18 \\ + & (-12x-6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

As in the previous example, both x and y dropped out of the equation, but we are left with an irrevocable contradiction, $0=-18$. This tells us that it is impossible to find a pair (x,y) which satisfies both equations; in other words, the system has no solution. Graphically, the lines $6x+3y=9$ and $4x+2y=12$ are distinct and parallel, so they do not intersect.

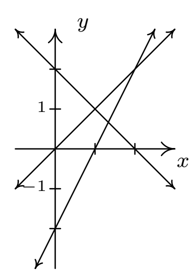
6. We can begin to solve our last system by adding the first two equations

$$\begin{array}{rrcr} & x-y & = & 0 \\ + & (x+y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x=1$. Substituting this into the first equation gives $1-y=0$ so that $y=1$. We seem to have determined a solution to our system, $(1,1)$. While this checks in the first two equations, when we substitute $x=1$ and $y=1$ into the third equation, we get $-2(1)+(1)=-2$ which simplifies to the contradiction $-1=-2$. Graphing the lines $x-y=0$, $x+y=2$, and $-2x+y=-2$, we see that the first two lines do, in fact, intersect at $(1,1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.



$6x+3y=9$
$\mathbf{4x+2y=12}$



$y-x=0$
$y+x=2$
$-2x+y=-2$

□

A few remarks about Example 8.1.1 are in order. It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish the two different types of behavior among consistent systems. Those which admit free variables are called **dependent**; those with no free variables are called **independent**.[4] Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 8.1.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, since we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called **underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

In order to move this section beyond a review of Intermediate Algebra, we now define what is meant by a linear equation in n variables.

Definition 8.2. A **linear equation in n variables**, x_1, x_2, $\ldots$, x_n, is an equation of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ where a_1, a_2, $\ldots a_n$ and c are real numbers and at least one of a_1, a_2, $\ldots$, a_n is nonzero.

Instead of using more familiar variables like x, y, and even z and/or w in Definition 8.2, we use subscripts to distinguish the different variables. We have no idea how many variables may be involved, so we use numbers to distinguish them instead of letters. (There is an endless supply of distinct numbers.) As an example, the linear equation $3x_1 - x_2 = 4$ represents the same relationship between the variables x_1 and x_2 as the equation $3x - y = 4$ does between the variables x and y. In addition, just as we cannot combine the terms in the expression $3x - y$, we cannot combine the terms in the expression $3x_1 - x_2$. Coupling more than one linear equation in n variables results in a **system of linear equations in n variables**. When solving these systems, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the kind of manipulations we've already employed. To this end, we first remind ourselves of the maneuvers which can be applied to a system of linear equations that result in an equivalent system.[9]

[4]In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5]The adjectives 'dependent' and 'independent' apply only to *consistent* systems – they describe the *type* of solutions. Is there a free variable (dependent) or not (independent)?

[6]If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

[7]We need more than two variables to give an example of the latter.

[8]Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.

[9]That is, a system with the same solution set.

Theorem 8.1. Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a]That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

We have seen plenty of instances of the second and third moves in Theorem 8.1 when we solved the systems in Example 8.1.1. The first move, while it obviously admits an equivalent system, seems silly. Our perception will change as we consider more equations and more variables in this, and later sections.

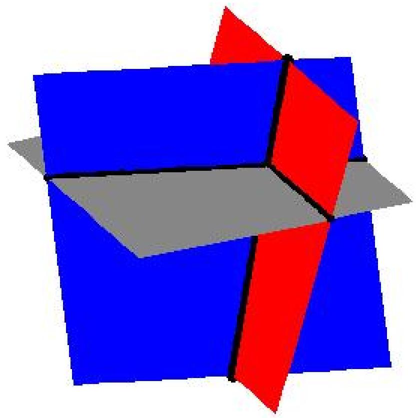
Consider the system of equations

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

Clearly $z = -1$, and we substitute this into the second equation $y - \frac{1}{2}(-1) = 4$ to obtain $y = \frac{7}{2}$. Finally, we substitute $y = \frac{7}{2}$ and $z = -1$ into the first equation to get $x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$, so that $x = \frac{8}{3}$. The reader can verify that these values of x, y and z satisfy all three original equations. It is tempting for us to write the solution to this system by extending the usual (x, y) notation to (x, y, z) and list our solution as $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$. The question quickly becomes what does an 'ordered triple' like $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space.[10] Moreover, just as equations involving the variables x and y describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving variables x, y, and z describe objects called **surfaces** in three-dimensional space. Each of the equations in the above system can be visualized as a plane situated in three-space. Geometrically, the system is trying to find the intersection, or common point, of all three planes. If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line,[11] so our intersection point is where all three of these lines meet.

[10]You were asked to think about this in Exercise 40 in Section 1.1.

[11]In fact, these lines are described by the parametric solutions to the systems formed by taking any two of these equations by themselves.



Since the geometry for equations involving more than two variables is complicated, we will focus our efforts on the algebra. Returning to the system

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

we note the reason it was so easy to solve is that the third equation is solved for z, the second equation involves only y and z, and since the coefficient of y is 1, it makes it easy to solve for y using our known value for z. Lastly, the coefficient of x in the first equation is 1 making it easy to substitute the known values of y and z and then solve for x. We formalize this pattern below for the most general systems of linear equations. Again, we use subscripted variables to describe the general case. The variable with the smallest subscript in a given equation is typically called the **leading variable** of that equation.

Definition 8.3. A system of linear equations with variables x_1, x_2, $\ldots x_n$ is said to be in **triangular form** provided all of the following conditions hold:

1. The subscripts of the variables in each equation are always increasing from left to right.
2. The leading variable in each equation has coefficient 1.
3. The subscript on the leading variable in a given equation is greater than the subscript on the leading variable in the equation above it.
4. Any equation without variables[a] cannot be placed above an equation with variables.

[a] necessarily an identity or contradiction

In our previous system, if we make the obvious choices $x = x_1$, $y = x_2$, and $z = x_3$, we see that the system is in triangular form.[12] An example of a more complicated system in triangular form is

$$\left\{ \begin{array}{rcr} x_1 - 4x_3 + x_4 - x_6 & = & 6 \\ x_2 + 2x_3 & = & 1 \\ x_4 + 3x_5 - x_6 & = & 8 \\ x_5 + 9x_6 & = & 10 \end{array} \right.$$

Our goal henceforth will be to transform a given system of linear equations into triangular form using the moves in Theorem 8.1.

Example 8.1.2. Use Theorem 8.1 to put the following systems into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

1. $\left\{ \begin{array}{rcr} 3x - y + z & = & 3 \\ 2x - 4y + 3z & = & 16 \\ x - y + z & = & 5 \end{array} \right.$
2. $\left\{ \begin{array}{rcr} 2x + 3y - z & = & 1 \\ 10x - z & = & 2 \\ 4x - 9y + 2z & = & 5 \end{array} \right.$
3. $\left\{ \begin{array}{rcr} 3x_1 + x_2 + x_4 & = & 6 \\ 2x_1 + x_2 - x_3 & = & 4 \\ x_2 - 3x_3 - 2x_4 & = & 0 \end{array} \right.$

Solution.

1. For definitiveness, we label the topmost equation in the system $E1$, the equation beneath that $E2$, and so forth. We now attempt to put the system in triangular form using an algorithm known as **Gaussian Elimination**. What this means is that, starting with x, we transform the system so that conditions 2 and 3 in Definition 8.3 are satisfied. Then we move on to the next variable, in this case y, and repeat. Since the variables in all of the equations have a consistent ordering from left to right, our first move is to get an x in $E1$'s spot with a coefficient of 1. While there are many ways to do this, the easiest is to apply the first move listed in Theorem 8.1 and interchange $E1$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & 3x - y + z & = & 3 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & x - y + z & = & 5 \end{array} \right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right.$$

To satisfy Definition 8.3, we need to eliminate the x's from $E2$ and $E3$. We accomplish this by replacing each of them with a sum of themselves and a multiple of $E1$. To eliminate the x from $E2$, we need to multiply $E1$ by -2 then add; to eliminate the x from $E3$, we need to multiply $E1$ by -3 then add. Applying the third move listed in Theorem 8.1 twice, we get

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & 2x - 4y + 3z & = & 16 \\ (E3) & 3x - y + z & = & 3 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -3E1 + E3]{\text{Replace } E2 \text{ with } -2E1 + E2} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & -2y + z & = & 6 \\ (E3) & 2y - 2z & = & -12 \end{array} \right.$$

[12] If letters are used instead of subscripted variables, Definition 8.3 can be suitably modified using alphabetical order of the variables instead of numerical order on the subscripts of the variables.

Now we enforce the conditions stated in Definition 8.3 for the variable y. To that end we need to get the coefficient of y in $E2$ equal to 1. We apply the second move listed in Theorem 8.1 and replace $E2$ with itself times $-\frac{1}{2}$.

$$\left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & -2y+z & = & 6 \\ (E3) & 2y-2z & = & -12 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{2}E2} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 2y-2z & = & -12 \end{array} \right.$$

To eliminate the y in $E3$, we add $-2E2$ to it.

$$\left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 2y-2z & = & -12 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } -2E2+E3} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array} \right.$$

Finally, we apply the second move from Theorem 8.1 one last time and multiply $E3$ by -1 to satisfy the conditions of Definition 8.3 for the variable z.

$$\left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } -1E3} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & z & = & 6 \end{array} \right.$$

Now we proceed to substitute. Plugging in $z=6$ into $E2$ gives $y-3=-3$ so that $y=0$. With $y=0$ and $z=6$, $E1$ becomes $x-0+6=5$, or $x=-1$. Our solution is $(-1,0,6)$. We leave it to the reader to check that substituting the respective values for x, y, and z into the original system results in three identities. Since we have found a solution, the system is consistent; since there are no free variables, it is independent.

2. Proceeding as we did in 1, our first step is to get an equation with x in the $E1$ position with 1 as its coefficient. Since there is no easy fix, we multiply $E1$ by $\frac{1}{2}$.

$$\left\{ \begin{array}{lrcr} (E1) & 2x+3y-z & = & 1 \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array} \right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{2}E1} \left\{ \begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array} \right.$$

Now it's time to take care of the x's in $E2$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x-z & = & 2 \\ (E3) & 4x-9y+2z & = & 5 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1+E3]{\text{Replace } E2 \text{ with } -10E1+E2} \left\{ \begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array} \right.$$

Our next step is to get the coefficient of y in $E2$ equal to 1. To that end, we have

$$\left\{\begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{15}E2} \left\{\begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y - \frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y + 4z & = & 3 \end{array}\right.$$

Finally, we rid $E3$ of y.

$$\left\{\begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y - \frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y + 4z & = & 3 \end{array}\right. \xrightarrow{\text{Replace } E3 \text{ with } 15E2 + E3} \left\{\begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 0 & = & 6 \end{array}\right.$$

The last equation, $0 = 6$, is a contradiction so the system has no solution. According to Theorem 8.1, since this system has no solutions, neither does the original, thus we have an inconsistent system.

3. For our last system, we begin by multiplying $E1$ by $\frac{1}{3}$ to get a coefficient of 1 on x_1.

$$\left\{\begin{array}{lrcr} (E1) & 3x_1 + x_2 + x_4 & = & 6 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$$

Next we eliminate x_1 from $E2$

$$\left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -2E1 + E2]{\text{Replace } E2} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$$

We switch $E2$ and $E3$ to get a coefficient of 1 for x_2.

$$\left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Switch } E2 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array}\right.$$

Finally, we eliminate x_2 in $E3$.

$$\left\{\begin{array}{lrcl} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & x_2-3x_3-2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2-x_3-\frac{2}{3}x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -\frac{1}{3}E2+E3]{\text{Replace } E3} \left\{\begin{array}{lrcl} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & x_2-3x_3-2x_4 & = & 0 \\ (E3) & 0 & = & 0 \end{array}\right.$$

Equation $E3$ reduces to $0 = 0$,which is always true. Since we have no equations with x_3 or x_4 as leading variables, they are both free, which means we have a consistent dependent system. We parametrize the solution set by letting $x_3 = s$ and $x_4 = t$ and obtain from $E2$ that $x_2 = 3s + 2t$. Substituting this and $x_4 = t$ into $E1$, we have $x_1 + \frac{1}{3}(3s + 2t) + \frac{1}{3}t = 2$ which gives $x_1 = 2-s-t$. Our solution is the set $\{(2-s-t, 2s+3t, s, t) \,|\, -\infty < s, t < \infty\}$.[13] We leave it to the reader to verify that the substitutions $x_1 = 2 - s - t$, $x_2 = 3s + 2t$, $x_3 = s$ and $x_4 = t$ satisfy the equations in the original system. □

Like all algorithms, Gaussian Elimination has the advantage of always producing what we need, but it can also be inefficient at times. For example, when solving 2 above, it is clear after we eliminated the x's in the second step to get the system

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array}\right.$$

that equations $E2$ and $E3$ when taken together form a contradiction since we have identical left hand sides and different right hand sides. The algorithm takes two more steps to reach this contradiction. We also note that substitution in Gaussian Elimination is delayed until all the elimination is done, thus it gets called **back-substitution**. This may also be inefficient in many cases. Rest assured, the technique of substitution as you may have learned it in Intermediate Algebra will once again take center stage in Section 8.7. Lastly, we note that the system in 3 above is underdetermined, and as it is consistent, we have free variables in our answer. We close this section with a standard 'mixture' type application of systems of linear equations.

Example 8.1.3. Lucas needs to create a 500 milliliters (mL) of a 40% acid solution. He has stock solutions of 30% and 90% acid as well as all of the distilled water he wants. Set-up and solve a system of linear equations which determines all of the possible combinations of the stock solutions and water which would produce the required solution.

Solution. We are after three unknowns, the amount (in mL) of the 30% stock solution (which we'll call x), the amount (in mL) of the 90% stock solution (which we'll call y) and the amount (in mL) of water (which we'll call w). We now need to determine some relationships between these variables. Our goal is to produce 500 milliliters of a 40% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 40% acid. We take each

[13]Here, any choice of s and t will determine a solution which is a point in 4-dimensional space. Yeah, we have trouble visualizing that, too.

of these qualities in turn. First, the total volume of 500 mL must be the sum of the contributed volumes of the two stock solutions and the water. That is

$$\text{amount of 30\% stock solution} + \text{amount of 90\% stock solution} + \text{amount of water} = 500\,\text{mL}$$

Using our defined variables, this reduces to $x + y + w = 500$. Next, we need to make sure the final solution is 40% acid. Since water contains no acid, the acid will come from the stock solutions only. We find 40% of 500 mL to be 200 mL which means the final solution must contain 200 mL of acid. We have

$$\text{amount of acid in 30\% stock solution} + \text{amount of acid 90\% stock solution} = 200\,\text{mL}$$

The amount of acid in x mL of 30% stock is $0.30x$ and the amount of acid in y mL of 90% solution is $0.90y$. We have $0.30x + 0.90y = 200$. Converting to fractions,[14] our system of equations becomes

$$\left\{ \begin{array}{rcl} x + y + w & = & 500 \\ \frac{3}{10}x + \frac{9}{10}y & = & 200 \end{array} \right.$$

We first eliminate the x from the second equation

$$\left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{10}x + \frac{9}{10}y & = & 200 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{3}{10}E1 + E2} \left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w & = & 50 \end{array} \right.$$

Next, we get a coefficient of 1 on the leading variable in $E2$

$$\left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & \frac{3}{5}y - \frac{3}{10}w & = & 50 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } \frac{5}{3}E2} \left\{ \begin{array}{lrcl} (E1) & x + y + w & = & 500 \\ (E2) & y - \frac{1}{2}w & = & \frac{250}{3} \end{array} \right.$$

Notice that we have no equation to determine w, and as such, w is free. We set $w = t$ and from $E2$ get $y = \frac{1}{2}t + \frac{250}{3}$. Substituting into $E1$ gives $x + \left(\frac{1}{2}t + \frac{250}{3}\right) + t = 500$ so that $x = -\frac{3}{2}t + \frac{1250}{3}$. This system is consistent, dependent and its solution set is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid -\infty < t < \infty\right\}$. While this answer checks algebraically, we have neglected to take into account that x, y and w, being amounts of acid and water, need to be nonnegative. That is, $x \geq 0$, $y \geq 0$ and $w \geq 0$. The constraint $x \geq 0$ gives us $-\frac{3}{2}t + \frac{1250}{3} \geq 0$, or $t \leq \frac{2500}{9}$. From $y \geq 0$, we get $\frac{1}{2}t + \frac{250}{3} \geq 0$ or $t \geq -\frac{500}{3}$. The condition $z \geq 0$ yields $t \geq 0$, and we see that when we take the set theoretic intersection of these intervals, we get $0 \leq t \leq \frac{2500}{9}$. Our final answer is $\left\{\left(-\frac{3}{2}t + \frac{1250}{3}, \frac{1}{2}t + \frac{250}{3}, t\right) \mid 0 \leq t \leq \frac{2500}{9}\right\}$. Of what practical use is our answer? Suppose there is only 100 mL of the 90% solution remaining and it is due to expire. Can we use all of it to make our required solution? We would have $y = 100$ so that $\frac{1}{2}t + \frac{250}{3} = 100$, and we get $t = \frac{100}{3}$. This means the amount of 30% solution required is $x = -\frac{3}{2}t + \frac{1250}{3} = -\frac{3}{2}\left(\frac{100}{3}\right) + \frac{1250}{3} = \frac{1100}{3}$ mL, and for the water, $w = t = \frac{100}{3}$ mL. The reader is invited to check that mixing these three amounts of our constituent solutions produces the required 40% acid mix. $\square$

[14] We do this only because we believe students can use all of the practice with fractions they can get!

8.1.1 Exercises

(Review Exercises) In Exercises 1 - 8, take a trip down memory lane and solve the given system using substitution and/or elimination. Classify each system as consistent independent, consistent dependent, or inconsistent. Check your answers both algebraically and graphically.

1. $\left\{\begin{array}{rcl} x+2y & = & 5 \\ x & = & 6 \end{array}\right.$

2. $\left\{\begin{array}{rcr} 2y-3x & = & 1 \\ y & = & -3 \end{array}\right.$

3. $\left\{\begin{array}{rcr} \frac{x+2y}{4} & = & -5 \\ \frac{3x-y}{2} & = & 1 \end{array}\right.$

4. $\left\{\begin{array}{rcl} \frac{2}{3}x-\frac{1}{5}y & = & 3 \\ \frac{1}{2}x+\frac{3}{4}y & = & 1 \end{array}\right.$

5. $\left\{\begin{array}{rcr} \frac{1}{2}x-\frac{1}{3}y & = & -1 \\ 2y-3x & = & 6 \end{array}\right.$

6. $\left\{\begin{array}{rcl} x+4y & = & 6 \\ \frac{1}{12}x+\frac{1}{3}y & = & \frac{1}{2} \end{array}\right.$

7. $\left\{\begin{array}{rcr} 3y-\frac{3}{2}x & = & -\frac{15}{2} \\ \frac{1}{2}x-y & = & \frac{3}{2} \end{array}\right.$

8. $\left\{\begin{array}{rcr} \frac{5}{6}x+\frac{5}{3}y & = & -\frac{7}{3} \\ -\frac{10}{3}x-\frac{20}{3}y & = & 10 \end{array}\right.$

In Exercises 9 - 26, put each system of linear equations into triangular form and solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

9. $\left\{\begin{array}{rcr} -5x+y & = & 17 \\ x+y & = & 5 \end{array}\right.$

10. $\left\{\begin{array}{rcl} x+y+z & = & 3 \\ 2x-y+z & = & 0 \\ -3x+5y+7z & = & 7 \end{array}\right.$

11. $\left\{\begin{array}{rcr} 4x-y+z & = & 5 \\ 2y+6z & = & 30 \\ x+z & = & 5 \end{array}\right.$

12. $\left\{\begin{array}{rcr} 4x-y+z & = & 5 \\ 2y+6z & = & 30 \\ x+z & = & 6 \end{array}\right.$

13. $\left\{\begin{array}{rcr} x+y+z & = & -17 \\ y-3z & = & 0 \end{array}\right.$

14. $\left\{\begin{array}{rcr} x-2y+3z & = & 7 \\ -3x+y+2z & = & -5 \\ 2x+2y+z & = & 3 \end{array}\right.$

15. $\left\{\begin{array}{rcr} 3x-2y+z & = & -5 \\ x+3y-z & = & 12 \\ x+y+2z & = & 0 \end{array}\right.$

16. $\left\{\begin{array}{rcr} 2x-y+z & = & -1 \\ 4x+3y+5z & = & 1 \\ 5y+3z & = & 4 \end{array}\right.$

17. $\left\{\begin{array}{rcr} x-y+z & = & -4 \\ -3x+2y+4z & = & -5 \\ x-5y+2z & = & -18 \end{array}\right.$

18. $\left\{\begin{array}{rcr} 2x-4y+z & = & -7 \\ x-2y+2z & = & -2 \\ -x+4y-2z & = & 3 \end{array}\right.$

19. $\left\{\begin{array}{rcl} 2x-y+z & = & 1 \\ 2x+2y-z & = & 1 \\ 3x+6y+4z & = & 9 \end{array}\right.$

20. $\left\{\begin{array}{rcr} x-3y-4z & = & 3 \\ 3x+4y-z & = & 13 \\ 2x-19y-19z & = & 2 \end{array}\right.$

21. $\left\{\begin{array}{rcr} x+y+z & = & 4 \\ 2x-4y-z & = & -1 \\ x-y & = & 2 \end{array}\right.$

22. $\left\{\begin{array}{rcr} x-y+z & = & 8 \\ 3x+3y-9z & = & -6 \\ 7x-2y+5z & = & 39 \end{array}\right.$

23. $\left\{\begin{array}{rcr} 2x-3y+z & = & -1 \\ 4x-4y+4z & = & -13 \\ 6x-5y+7z & = & -25 \end{array}\right.$

24. $\left\{\begin{array}{rcr} 2x_1+x_2-12x_3-x_4 & = & 16 \\ -x_1+x_2+12x_3-4x_4 & = & -5 \\ 3x_1+2x_2-16x_3-3x_4 & = & 25 \\ x_1+2x_2-5x_4 & = & 11 \end{array}\right.$

25. $\left\{\begin{array}{rcr} x_1-x_3 & = & -2 \\ 2x_2-x_4 & = & 0 \\ x_1-2x_2+x_3 & = & 0 \\ -x_3+x_4 & = & 1 \end{array}\right.$

26. $\left\{\begin{array}{rcr} x_1-x_2-5x_3+3x_4 & = & -1 \\ x_1+x_2+5x_3-3x_4 & = & 0 \\ x_2+5x_3-3x_4 & = & 1 \\ x_1-2x_2-10x_3+6x_4 & = & -1 \end{array}\right.$

27. Find two other forms of the parametric solution to Exercise 11 above by reorganizing the equations so that x or y can be the free variable.

28. A local buffet charges \$7.50 per person for the basic buffet and \$9.25 for the deluxe buffet (which includes crab legs.) If 27 diners went out to eat and the total bill was \$227.00 before taxes, how many chose the basic buffet and how many chose the deluxe buffet?

29. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

30. Skippy has a total of \$10,000 to split between two investments. One account offers 3% simple interest, and the other account offers 8% simple interest. For tax reasons, he can only earn \$500 in interest the entire year. How much money should Skippy invest in each account to earn \$500 in interest for the year?

31. A 10% salt solution is to be mixed with pure water to produce 75 gallons of a 3% salt solution. How much of each are needed?

32. At The Crispy Critter's Head Shop and Patchouli Emporium along with their dried up weeds, sunflower seeds and astrological postcards they sell an herbal tea blend. By weight, Type I herbal tea is 30% peppermint, 40% rose hips and 30% chamomile, Type II has percents 40%, 20% and 40%, respectively, and Type III has percents 35%, 30% and 35%, respectively. How much of each Type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips and chamomile?

33. Discuss with your classmates how you would approach Exercise 32 above if they needed to use up a pound of Type I tea to make room on the shelf for a new canister.

34. If you were to try to make 100 mL of a 60% acid solution using stock solutions at 20% and 40%, respectively, what would the triangular form of the resulting system look like? Explain.

8.1.2 Answers

1. Consistent independent
   Solution $\left(6, -\frac{1}{2}\right)$

2. Consistent independent
   Solution $\left(-\frac{7}{3}, -3\right)$

3. Consistent independent
   Solution $\left(-\frac{16}{7}, -\frac{62}{7}\right)$

4. Consistent independent
   Solution $\left(\frac{49}{12}, -\frac{25}{18}\right)$

5. Consistent dependent
   Solution $\left(t, \frac{3}{2}t + 3\right)$
   for all real numbers t

6. Consistent dependent
   Solution $(6 - 4t, t)$
   for all real numbers t

7. Inconsistent
   No solution

8. Inconsistent
   No solution

Because triangular form is not unique, we give only one possible answer to that part of the question. Yours may be different and still be correct.

9. $\left\{ \begin{array}{rcr} x + y & = & 5 \\ y & = & 7 \end{array} \right.$

   Consistent independent
   Solution $(-2, 7)$

10. $\left\{ \begin{array}{rcr} x - \frac{5}{3}y - \frac{7}{3}z & = & -\frac{7}{3} \\ y + \frac{5}{4}z & = & 2 \\ z & = & 0 \end{array} \right.$

    Consistent independent
    Solution $(1, 2, 0)$

11. $\left\{ \begin{array}{rcr} x - \frac{1}{4}y + \frac{1}{4}z & = & \frac{5}{4} \\ y + 3z & = & 15 \\ 0 & = & 0 \end{array} \right.$

    Consistent dependent
    Solution $(-t + 5, -3t + 15, t)$
    for all real numbers t

12. $\left\{ \begin{array}{rcr} x - \frac{1}{4}y + \frac{1}{4}z & = & \frac{5}{4} \\ y + 3z & = & 15 \\ 0 & = & 1 \end{array} \right.$

    Inconsistent
    No solution

13. $\left\{ \begin{array}{rcr} x + y + z & = & -17 \\ y - 3z & = & 0 \end{array} \right.$

    Consistent dependent
    Solution $(-4t - 17, 3t, t)$
    for all real numbers t

14. $\left\{ \begin{array}{rcr} x - 2y + 3z & = & 7 \\ y - \frac{11}{5}z & = & -\frac{16}{5} \\ z & = & 1 \end{array} \right.$

    Consistent independent
    Solution $(2, -1, 1)$

15. $\left\{\begin{array}{rcr} x+y+2z & = & 0 \\ y-\frac{3}{2}z & = & 6 \\ z & = & -2 \end{array}\right.$

Consistent independent
Solution $(1,3,-2)$

16. $\left\{\begin{array}{rcr} x-\frac{1}{2}y+\frac{1}{2}z & = & -\frac{1}{2} \\ y+\frac{3}{5}z & = & \frac{3}{5} \\ 0 & = & 1 \end{array}\right.$

Inconsistent
no solution

17. $\left\{\begin{array}{rcr} x-y+z & = & -4 \\ y-7z & = & 17 \\ z & = & -2 \end{array}\right.$

Consistent independent
Solution $(1,3,-2)$

18. $\left\{\begin{array}{rcr} x-2y+2z & = & -2 \\ y & = & \frac{1}{2} \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $\left(-3,\frac{1}{2},1\right)$

19. $\left\{\begin{array}{rcr} x-\frac{1}{2}y+\frac{1}{2}z & = & \frac{1}{2} \\ y-\frac{2}{3}z & = & 0 \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $\left(\frac{1}{3},\frac{2}{3},1\right)$

20. $\left\{\begin{array}{rcr} x-3y-4z & = & 3 \\ y+\frac{11}{13}z & = & \frac{4}{13} \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $\left(\frac{19}{13}t+\frac{51}{13},-\frac{11}{13}t+\frac{4}{13},t\right)$
for all real numbers t

21. $\left\{\begin{array}{rcr} x+y+z & = & 4 \\ y+\frac{1}{2}z & = & \frac{3}{2} \\ 0 & = & 1 \end{array}\right.$

Inconsistent
no solution

22. $\left\{\begin{array}{rcr} x-y+z & = & 8 \\ y-2z & = & -5 \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $(4,-3,1)$

23. $\left\{\begin{array}{rcr} x-\frac{3}{2}y+\frac{1}{2}z & = & -\frac{1}{2} \\ y+z & = & -\frac{11}{2} \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $\left(-2t-\frac{35}{4},-t-\frac{11}{2},t\right)$
for all real numbers t

24. $\left\{\begin{array}{rcr} x_1+\frac{2}{3}x_2-\frac{16}{3}x_3-x_4 & = & \frac{25}{3} \\ x_2+4x_3-3x_4 & = & 2 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $(8s-t+7,-4s+3t+2,s,t)$
for all real numbers s and t

25. $\left\{\begin{array}{rcr} x_1-x_3 & = & -2 \\ x_2-\frac{1}{2}x_4 & = & 0 \\ x_3-\frac{1}{2}x_4 & = & 1 \\ x_4 & = & 4 \end{array}\right.$

Consistent independent
Solution $(1,2,3,4)$

26. $\left\{ \begin{array}{rcr} x_1 - x_2 - 5x_3 + 3x_4 & = & -1 \\ x_2 + 5x_3 - 3x_4 & = & \frac{1}{2} \\ 0 & = & 1 \\ 0 & = & 0 \end{array} \right.$ Inconsistent
No solution

27. If x is the free variable then the solution is $(t, 3t, -t+5)$ and if y is the free variable then the solution is $\left(\frac{1}{3}t, t, -\frac{1}{3}t+5\right)$.

28. 13 chose the basic buffet and 14 chose the deluxe buffet.

29. Mavis needs 20 pounds of \$3 per pound coffee and 30 pounds of \$8 per pound coffee.

30. Skippy needs to invest \$6000 in the 3% account and \$4000 in the 8% account.

31. 22.5 gallons of the 10% solution and 52.5 gallons of pure water.

32. $\frac{4}{3} - \frac{1}{2}t$ pounds of Type I, $\frac{2}{3} - \frac{1}{2}t$ pounds of Type II and t pounds of Type III where $0 \leq t \leq \frac{4}{3}$.

8.2 Systems of Linear Equations: Augmented Matrices

In Section 8.1 we introduced Gaussian Elimination as a means of transforming a system of linear equations into triangular form with the ultimate goal of producing an equivalent system of linear equations which is easier to solve. If we take a step back and study the process, we see that all of our moves are determined entirely by the *coefficients* of the variables involved, and not the variables themselves. Much the same thing happened when we studied long division in Section 3.2. Just as we developed synthetic division to streamline that process, in this section, we introduce a similar bookkeeping device to help us solve systems of linear equations. To that end, we define a **matrix** as a rectangular array of real numbers. We typically enclose matrices with square brackets, '[' and ']', and we size matrices by the number of rows and columns they have. For example, the **size** (sometimes called the **dimension**) of

$$\left[\begin{array}{rrr} 3 & 0 & -1 \\ 2 & -5 & 10 \end{array}\right]$$

is 2×3 because it has 2 rows and 3 columns. The individual numbers in a matrix are called its **entries** and are usually labeled with double subscripts: the first tells which row the element is in and the second tells which column it is in. The rows are numbered from top to bottom and the columns are numbered from left to right. Matrices themselves are usually denoted by uppercase letters (A, B, C, etc.) while their entries are usually denoted by the corresponding letter. So, for instance, if we have

$$A = \left[\begin{array}{rrr} 3 & 0 & -1 \\ 2 & -5 & 10 \end{array}\right]$$

then $a_{11} = 3$, $a_{12} = 0$, $a_{13} = -1$, $a_{21} = 2$, $a_{22} = -5$, and $a_{23} = 10$. We shall explore matrices as mathematical objects with their own algebra in Section 8.3 and introduce them here solely as a bookkeeping device. Consider the system of linear equations from number 2 in Example 8.1.2

$$\left\{ \begin{array}{lrcl} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right.$$

We encode this system into a matrix by assigning each equation to a corresponding row. Within that row, each variable and the constant gets its own column, and to separate the variables on the left hand side of the equation from the constants on the right hand side, we use a vertical bar, $|$. Note that in $E2$, since y is not present, we record its coefficient as 0. The matrix associated with this system is

$$\begin{array}{c} \\ (E1) \rightarrow \\ (E2) \rightarrow \\ (E3) \rightarrow \end{array} \begin{array}{c} \begin{array}{rrrr} x \quad & y \quad & z \quad & c \end{array} \\ \left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array}\right] \end{array}$$

This matrix is called an **augmented matrix** because the column containing the constants is appended to the matrix containing the coefficients.[1] To solve this system, we can use the same kind operations on the *rows* of the matrix that we performed on the *equations* of the system. More specifically, we have the following analog of Theorem 8.1 below.

Theorem 8.2. Row Operations: Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[a]
- Replace a row with itself plus a nonzero multiple of another row.[b]

[a] That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b] Where we add entries in corresponding columns.

As a demonstration of the moves in Theorem 8.2, we revisit some of the steps that were used in solving the systems of linear equations in Example 8.1.2 of Section 8.1. The reader is encouraged to perform the indicated operations on the rows of the augmented matrix to see that the machinations are identical to what is done to the coefficients of the variables in the equations. We first see a demonstration of switching two rows using the first step of part 1 in Example 8.1.2.

$$\left\{\begin{array}{lrcr} (E1) & 3x-y+z & = & 3 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & x-y+z & = & 5 \end{array}\right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{\begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & 3x-y+z & = & 3 \end{array}\right.$$

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right] \xrightarrow{\text{Switch } R1 \text{ and } R3} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 5 \\ 2 & -4 & 3 & 16 \\ 3 & -1 & 1 & 3 \end{array}\right]$$

Next, we have a demonstration of replacing a row with a nonzero multiple of itself using the first step of part 3 in Example 8.1.2.

$$\left\{\begin{array}{lrcr} (E1) & 3x_1+x_2+x_4 & = & 6 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcr} (E1) & x_1+\frac{1}{3}x_2+\frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1+x_2-x_3 & = & 4 \\ (E3) & x_2-3x_3-2x_4 & = & 0 \end{array}\right.$$

$$\left[\begin{array}{rrrr|r} 3 & 1 & 0 & 1 & 6 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R1} \left[\begin{array}{rrrr|r} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 2 \\ 2 & 1 & -1 & 0 & 4 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right]$$

Finally, we have an example of replacing a row with itself plus a multiple of another row using the second step from part 2 in Example 8.1.2.

[1] We shall study the coefficient and constant matrices separately in Section 8.3.

$$\left\{ \begin{array}{rrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1 + E3]{\text{Replace } E2 \text{ with } -10E1 + E2} \left\{ \begin{array}{rrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array} \right.$$

$$\left[\begin{array}{rrr|r} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 10 & 0 & -1 & 2 \\ 4 & -9 & 2 & 5 \end{array} \right] \xrightarrow[\text{Replace } R3 \text{ with } -4R1 + R3]{\text{Replace } R2 \text{ with } -10R1 + R2} \left[\begin{array}{rrr|r} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -15 & 4 & -3 \\ 0 & -15 & 4 & 3 \end{array} \right]$$

The matrix equivalent of 'triangular form' is **row echelon form**. The reader is encouraged to refer to Definition 8.3 for comparison. Note that the analog of 'leading variable' of an equation is 'leading entry' of a row. Specifically, the first nonzero entry (if it exists) in a row is called the **leading entry** of that row.

Definition 8.4. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

To solve a system of a linear equations using an augmented matrix, we encode the system into an augmented matrix and apply Gaussian Elimination to the rows to get the matrix into row-echelon form. We then decode the matrix and back substitute. The next example illustrates this nicely.

Example 8.2.1. Use an augmented matrix to transform the following system of linear equations into triangular form. Solve the system.

$$\left\{ \begin{array}{rcr} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{array} \right.$$

Solution. We first encode the system into an augmented matrix.

$$\left\{ \begin{array}{rcr} 3x - y + z & = & 8 \\ x + 2y - z & = & 4 \\ 2x + 3y - 4z & = & 10 \end{array} \right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array} \right]$$

Thinking back to Gaussian Elimination at an equations level, our first order of business is to get x in $E1$ with a coefficient of 1. At the matrix level, this means getting a leading 1 in $R1$. This is in accordance with the first criteria in Definition 8.4. To that end, we interchange $R1$ and $R2$.

$$\left[\begin{array}{rrr|r} 3 & -1 & 1 & 8 \\ 1 & 2 & -1 & 4 \\ 2 & 3 & -4 & 10 \end{array} \right] \xrightarrow{\text{Switch } R1 \text{ and } R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array} \right]$$

Our next step is to eliminate the x's from $E2$ and $E3$. From a matrix standpoint, this means we need 0's below the leading 1 in $R1$. This guarantees the leading 1 in $R2$ will be to the right of the leading 1 in $R1$ in accordance with the second requirement of Definition 8.4.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & -1 & 1 & 8 \\ 2 & 3 & -4 & 10 \end{array}\right] \xrightarrow[\text{Replace } R3 \text{ with } -2R1+R3]{\text{Replace } R2 \text{ with } -3R1+R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right]$$

Now we repeat the above process for the variable y which means we need to get the leading entry in $R2$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & -7 & 4 & -4 \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -\frac{1}{7}R2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right]$$

To guarantee the leading 1 in $R3$ is to the right of the leading 1 in $R2$, we get a 0 in the second column of $R3$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & -1 & -2 & 2 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } R2+R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right]$$

Finally, we get the leading entry in $R3$ to be 1.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & -\frac{18}{7} & \frac{18}{7} \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{7}{18}R3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Decoding from the matrix gives a system in triangular form

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{ \begin{array}{rcr} x+2y-z & = & 4 \\ y-\frac{4}{7}z & = & \frac{4}{7} \\ z & = & -1 \end{array} \right.$$

We get $z=-1$, $y=\frac{4}{7}z+\frac{4}{7}=\frac{4}{7}(-1)+\frac{4}{7}=0$ and $x=-2y+z+4=-2(0)+(-1)+4=3$ for a final answer of $(3,0,-1)$. We leave it to the reader to check. □

As part of Gaussian Elimination, we used row operations to obtain 0's beneath each leading 1 to put the matrix into row echelon form. If we also require that 0's are the only numbers above a leading 1, we have what is known as the **reduced row echelon form** of the matrix.

Definition 8.5. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.
2. The leading 1s are the only nonzero entry in their respective columns.

Of what significance is the reduced row echelon form of a matrix? To illustrate, let's take the row echelon form from Example 8.2.1 and perform the necessary steps to put into reduced row echelon form. We start by using the leading 1 in $R3$ to zero out the numbers in the rows above it.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{4}{7} & \frac{4}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[\text{Replace } R2 \text{ with } \frac{4}{7}R3+R2]{\text{Replace } R1 \text{ with } R3+R1} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Finally, we take care of the 2 in $R1$ above the leading 1 in $R2$.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -2R2+R1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

To our surprise and delight, when we decode this matrix, we obtain the solution instantly without having to deal with any back-substitution at all.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{\begin{array}{rcr} x & = & 3 \\ y & = & 0 \\ z & = & -1 \end{array}\right.$$

Note that in the previous discussion, we could have started with $R2$ and used it to get a zero above its leading 1 and then done the same for the leading 1 in $R3$. By starting with $R3$, however, we get more zeros first, and the more zeros there are, the faster the remaining calculations will be.[2] It is also worth noting that while a matrix has several[3] row echelon forms, it has only one reduced row echelon form. The process by which we have put a matrix into reduced row echelon form is called **Gauss-Jordan Elimination**.

Example 8.2.2. Solve the following system using an augmented matrix. Use Gauss-Jordan Elimination to put the augmented matrix into reduced row echelon form.

$$\left\{\begin{array}{rcr} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{array}\right.$$

Solution. We first encode the system into a matrix. (Pay attention to the subscripts!)

$$\left\{\begin{array}{rcr} x_2 - 3x_1 + x_4 & = & 2 \\ 2x_1 + 4x_3 & = & 5 \\ 4x_2 - x_4 & = & 3 \end{array}\right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrrr|r} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

Next, we get a leading 1 in the first column of $R1$.

$$\left[\begin{array}{rrrr|r} -3 & 1 & 0 & 1 & 2 \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } -\frac{1}{3}R1} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

[2] Carl also finds starting with $R3$ to be more symmetric, in a purely poetic way.
[3] infinite, in fact

Now we eliminate the nonzero entry below our leading 1.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & 0 & 4 & 0 & 5 \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -2R1+R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We proceed to get a leading 1 in $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & 4 & \frac{2}{3} & \frac{19}{3} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } \frac{3}{2}R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right]$$

We now zero out the entry below the leading 1 in $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 4 & 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -4R2+R3} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right]$$

Next, it's time for a leading 1 in $R3$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & -24 & -5 & -35 \end{array}\right] \xrightarrow{\text{Replace } R3 \text{ with } -\frac{1}{24}R3} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

The matrix is now in row echelon form. To get the reduced row echelon form, we start with the last leading 1 we produced and work to get 0's above it.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 6 & 1 & \frac{19}{2} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R2 \text{ with } -6R3+R2} \left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

Lastly, we get a 0 above the leading 1 of $R2$.

$$\left[\begin{array}{rrrr|r} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Replace } R1 \text{ with } \frac{1}{3}R2+R1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right]$$

At last, we decode to get

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{5}{12} & -\frac{5}{12} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{35}{24} \end{array}\right] \xrightarrow{\text{Decode from the matrix}} \left\{ \begin{array}{rcr} x_1 - \frac{5}{12}x_4 & = & -\frac{5}{12} \\ x_2 - \frac{1}{4}x_4 & = & \frac{3}{4} \\ x_3 + \frac{5}{24}x_4 & = & \frac{35}{24} \end{array} \right.$$

We have that x_4 is free and we assign it the parameter t. We obtain $x_3 = -\frac{5}{24}t + \frac{35}{24}$, $x_2 = \frac{1}{4}t + \frac{3}{4}$, and $x_1 = \frac{5}{12}t - \frac{5}{12}$. Our solution is $\left\{ \left(\frac{5}{12}t - \frac{5}{12}, \frac{1}{4}t + \frac{3}{4}, -\frac{5}{24}t + \frac{35}{24}, t \right) : -\infty < t < \infty \right\}$ and leave it to the reader to check. $\square$

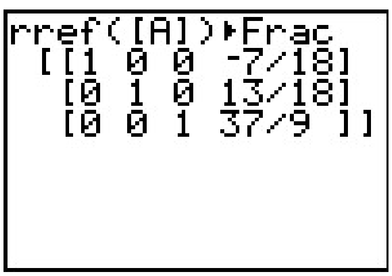
Like all good algorithms, putting a matrix in row echelon or reduced row echelon form can easily be programmed into a calculator, and, doubtless, your graphing calculator has such a feature. We use this in our next example.

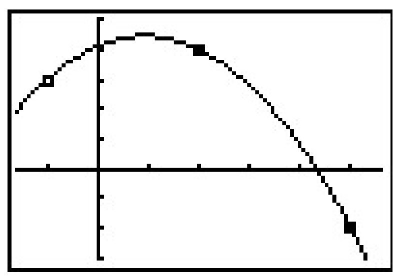
Example 8.2.3. Find the quadratic function passing through the points $(-1,3)$, $(2,4)$, $(5,-2)$.

Solution. According to Definition 2.5, a quadratic function has the form $f(x) = ax^2+bx+c$ where $a \neq 0$. Our goal is to find a, b and c so that the three given points are on the graph of f. If $(-1,3)$ is on the graph of f, then $f(-1) = 3$, or $a(-1)^2 + b(-1) + c = 3$ which reduces to $a - b + c = 3$, an honest-to-goodness linear equation with the variables a, b and c. Since the point $(2,4)$ is also on the graph of f, then $f(2) = 4$ which gives us the equation $4a + 2b + c = 4$. Lastly, the point $(5,-2)$ is on the graph of f gives us $25a+5b+c = -2$. Putting these together, we obtain a system of three linear equations. Encoding this into an augmented matrix produces

$$\left\{ \begin{array}{rcr} a - b + c & = & 3 \\ 4a + 2b + c & = & 4 \\ 25a + 5b + c & = & -2 \end{array} \right. \xrightarrow{\text{Encode into the matrix}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 4 & 2 & 1 & 4 \\ 25 & 5 & 1 & -2 \end{array} \right]$$

Using a calculator,[4] we find $a = -\frac{7}{18}$, $b = \frac{13}{18}$ and $c = \frac{37}{9}$. Hence, the one and only quadratic which fits the bill is $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$. To verify this analytically, we see that $f(-1) = 3$, $f(2) = 4$, and $f(5) = -2$. We can use the calculator to check our solution as well by plotting the three data points and the function f.





The graph of $f(x) = -\frac{7}{18}x^2 + \frac{13}{18}x + \frac{37}{9}$
with the points $(-1,3)$, $(2,4)$ and $(5,-2)$

$\square$

[4] We've tortured you enough already with fractions in this exposition!

8.2.1 Exercises

In Exercises 1 - 6, state whether the given matrix is in reduced row echelon form, row echelon form only or in neither of those forms.

1. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & -4 & 3 & 16 \\ 1 & -1 & 1 & 5 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

5. $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

6. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$

In Exercises 7 - 12, the following matrices are in reduced row echelon form. Determine the solution of the corresponding system of linear equations or state that the system is inconsistent.

7. $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 19 \end{array}\right]$

9. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 6 & -6 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right]$

10. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$

11. $\left[\begin{array}{cccc|c} 1 & 0 & -8 & 1 & 7 \\ 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

12. $\left[\begin{array}{ccc|c} 1 & 0 & 9 & -3 \\ 0 & 1 & -4 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 13 - 26, solve the following systems of linear equations using the techniques discussed in this section. Compare and contrast these techniques with those you used to solve the systems in the Exercises in Section 8.1.

13. $\left\{ \begin{array}{rcr} -5x + y & = & 17 \\ x + y & = & 5 \end{array} \right.$

14. $\left\{ \begin{array}{rcr} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

15. $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 5 \end{array} \right.$

16. $\left\{ \begin{array}{rcr} x - 2y + 3z & = & 7 \\ -3x + y + 2z & = & -5 \\ 2x + 2y + z & = & 3 \end{array} \right.$

17. $\left\{ \begin{array}{rcr} 3x - 2y + z & = & -5 \\ x + 3y - z & = & 12 \\ x + y + 2z & = & 0 \end{array} \right.$

18. $\left\{ \begin{array}{rcr} 2x - y + z & = & -1 \\ 4x + 3y + 5z & = & 1 \\ 5y + 3z & = & 4 \end{array} \right.$

19. $\left\{ \begin{array}{rcr} x - y + z & = & -4 \\ -3x + 2y + 4z & = & -5 \\ x - 5y + 2z & = & -18 \end{array} \right.$

20. $\left\{ \begin{array}{rcr} 2x - 4y + z & = & -7 \\ x - 2y + 2z & = & -2 \\ -x + 4y - 2z & = & 3 \end{array} \right.$

21. $\left\{\begin{array}{rcl} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{array}\right.$

22. $\left\{\begin{array}{rcr} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{array}\right.$

23. $\left\{\begin{array}{rcr} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{array}\right.$

24. $\left\{\begin{array}{rcr} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{array}\right.$

25. $\left\{\begin{array}{rcr} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{array}\right.$

26. $\left\{\begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array}\right.$

27. It's time for another meal at our local buffet. This time, 22 diners (5 of whom were children) feasted for \$162.25, before taxes. If the kids buffet is \$4.50, the basic buffet is \$7.50, and the deluxe buffet (with crab legs) is \$9.25, find out how many diners chose the deluxe buffet.

28. Carl wants to make a party mix consisting of almonds (which cost \$7 per pound), cashews (which cost \$5 per pound), and peanuts (which cost \$2 per pound.) If he wants to make a 10 pound mix with a budget of \$35, what are the possible combinations almonds, cashews, and peanuts? (You may find it helpful to review Example 8.1.3 in Section 8.1.)

29. Find the quadratic function passing through the points $(-2, 1)$, $(1, 4)$, $(3, -2)$

30. At 9 PM, the temperature was 60°F; at midnight, the temperature was 50°F; and at 6 AM, the temperature was 70°F . Use the technique in Example 8.2.3 to fit a quadratic function to these data with the temperature, T, measured in degrees Fahrenheit, as the dependent variable, and the number of hours after 9 PM, t, measured in hours, as the independent variable. What was the coldest temperature of the night? When did it occur?

31. The price for admission into the Stitz-Zeager Sasquatch Museum and Research Station is \$15 for adults and \$8 for kids 13 years old and younger. When the Zahlenreich family visits the museum their bill is \$38 and when the Nullsatz family visits their bill is \$39. One day both families went together and took an adult babysitter along to watch the kids and the total admission charge was \$92. Later that summer, the adults from both families went without the kids and the bill was \$45. Is that enough information to determine how many adults and children are in each family? If not, state whether the resulting system is inconsistent or consistent dependent. In the latter case, give at least two plausible solutions.

32. Use the technique in Example 8.2.3 to find the line between the points $(-3, 4)$ and $(6, 1)$. How does your answer compare to the slope-intercept form of the line in Equation 2.3?

33. With the help of your classmates, find at least two different row echelon forms for the matrix

$$\left[\begin{array}{rr|r} 1 & 2 & 3 \\ 4 & 12 & 8 \end{array}\right]$$

8.2.2 Answers

1. Reduced row echelon form
2. Neither
3. Row echelon form only
4. Reduced row echelon form
5. Reduced row echelon form
6. Row echelon form only
7. $(-2, 7)$
8. $(-3, 20, 19)$
9. $(-3t + 4, -6t - 6, 2, t)$ for all real numbers t
10. Inconsistent
11. $(8s - t + 7, -4s + 3t + 2, s, t)$ for all real numbers s and t
12. $(-9t - 3, 4t + 20, t)$ for all real numbers t
13. $(-2, 7)$
14. $(1, 2, 0)$
15. $(-t + 5, -3t + 15, t)$ for all real numbers t
16. $(2, -1, 1)$
17. $(1, 3, -2)$
18. Inconsistent
19. $(1, 3, -2)$
20. $\left(-3, \frac{1}{2}, 1\right)$
21. $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$
22. $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$ for all real numbers t
23. Inconsistent
24. $(4, -3, 1)$
25. $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$ for all real numbers t
26. $(1, 2, 3, 4)$
27. This time, 7 diners chose the deluxe buffet.
28. If t represents the amount (in pounds) of peanuts, then we need $1.5t - 7.5$ pounds of almonds and $17.5 - 2.5t$ pounds of cashews. Since we can't have a negative amount of nuts, $5 \leq t \leq 7$.
29. $f(x) = -\frac{4}{5}x^2 + \frac{1}{5}x + \frac{23}{5}$
30. $T(t) = \frac{20}{27}t^2 - \frac{50}{9}t + 60$. Lowest temperature of the evening $\frac{595}{12} \approx 49.58°\text{F}$ at 12:45 AM.

31. Let x_1 and x_2 be the numbers of adults and children, respectively, in the Zahlenreich family and let x_3 and x_4 be the numbers of adults and children, respectively, in the Nullsatz family. The system of equations determined by the given information is

$$\left\{ \begin{array}{rcl} 15x_1 + 8x_2 & = & 38 \\ 15x_3 + 8x_4 & = & 39 \\ 15x_1 + 8x_2 + 15x_3 + 8x_4 & = & 77 \\ 15x_1 + 15x_3 & = & 45 \end{array} \right.$$

We subtracted the cost of the babysitter in E3 so the constant is 77, not 92. This system is consistent dependent and its solution is $\left(\frac{8}{15}t + \frac{2}{5}, -t + 4, -\frac{8}{15}t + \frac{13}{5}, t\right)$. Our variables represent numbers of adults and children so they must be whole numbers. Running through the values $t = 0, 1, 2, 3, 4$ yields only one solution where all four variables are whole numbers; $t = 3$ gives us $(2, 1, 1, 3)$. Thus there are 2 adults and 1 child in the Zahlenreichs and 1 adult and 3 kids in the Nullsatzs.

8.3 Matrix Arithmetic

In Section 8.2, we used a special class of matrices, the augmented matrices, to assist us in solving systems of linear equations. In this section, we study matrices as mathematical objects of their own accord, temporarily divorced from systems of linear equations. To do so conveniently requires some more notation. When we write $A = [a_{ij}]_{m \times n}$, we mean A is an m by n matrix[1] and a_{ij} is the entry found in the ith row and jth column. Schematically, we have

$$A = \begin{array}{cc} \begin{array}{c} j \text{ counts columns} \\ \text{from left to right} \end{array} & \\ \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] & \begin{array}{c} i \text{ counts rows} \\ \text{from top to bottom} \end{array} \end{array}$$

With this new notation we can define what it means for two matrices to be equal.

Definition 8.6. Matrix Equality: Two matrices are said to be **equal** if they are the same size and their corresponding entries are equal. More specifically, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times r}$, we write $A = B$ provided

1. $m = p$ and $n = r$
2. $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$.

Essentially, two matrices are equal if they are the same size and they have the same numbers in the same spots.[2] For example, the two 2×3 matrices below are, despite appearances, equal.

$$\left[\begin{array}{rrr} 0 & -2 & 9 \\ 25 & 117 & -3 \end{array}\right] = \left[\begin{array}{ccc} \ln(1) & \sqrt[3]{-8} & e^{2\ln(3)} \\ 125^{2/3} & 3^2 \cdot 13 & \log(0.001) \end{array}\right]$$

Now that we have an agreed upon understanding of what it means for two matrices to equal each other, we may begin defining arithmetic operations on matrices. Our first operation is addition.

Definition 8.7. Matrix Addition: Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices. More specifically, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, we define

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$

As an example, consider the sum below.

[1] Recall that means A has m rows and n columns.

[2] Critics may well ask: Why not leave it at that? Why the need for all the notation in Definition 8.6? It is the authors' attempt to expose you to the wonderful world of mathematical precision.

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ -5 & -3 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 3+4 \\ 4+(-5) & (-1)+(-3) \\ 0+8 & (-7)+1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -1 & -4 \\ 8 & -6 \end{bmatrix}$$

It is worth the reader's time to think what would have happened had we reversed the order of the summands above. As we would expect, we arrive at the same answer. In general, $A + B = B + A$ for matrices A and B, provided they are the same size so that the sum is defined in the first place. This is the **commutative property** of matrix addition. To see why this is true in general, we appeal to the definition of matrix addition. Given $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$,

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n} = [b_{ij} + a_{ij}]_{m\times n} = [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n} = B + A$$

where the second equality is the definition of $A + B$, the third equality holds by the commutative law of real number addition, and the fourth equality is the definition of $B + A$. In other words, matrix addition is commutative because real number addition is. A similar argument shows the **associative property** of matrix addition also holds, inherited in turn from the associative law of real number addition. Specifically, for matrices A, B, and C of the same size, $(A + B) + C = A + (B + C)$. In other words, when adding more than two matrices, it doesn't matter how they are grouped. This means that we can write $A + B + C$ without parentheses and there is no ambiguity as to what this means.[3] These properties and more are summarized in the following theorem.

Theorem 8.3. Properties of Matrix Addition

- **Commutative Property:** For all $m \times n$ matrices, $A + B = B + A$
- **Associative Property:** For all $m \times n$ matrices, $(A + B) + C = A + (B + C)$
- **Identity Property:** If $0_{m\times n}$ is the $m \times n$ matrix whose entries are all 0, then $0_{m\times n}$ is called the $\boldsymbol{m \times n}$ **additive identity** and for all $m \times n$ matrices A

$$A + 0_{m\times n} = 0_{m\times n} + A = A$$

- **Inverse Property:** For every given $m \times n$ matrix A, there is a unique matrix denoted $-A$ called the **additive inverse of** $\boldsymbol{A}$ such that

$$A + (-A) = (-A) + A = 0_{m\times n}$$

The identity property is easily verified by resorting to the definition of matrix addition; just as the number 0 is the additive identity for real numbers, the matrix comprised of all 0's does the same job for matrices. To establish the inverse property, given a matrix $A = [a_{ij}]_{m\times n}$, we are looking for a matrix $B = [b_{ij}]_{m\times n}$ so that $A + B = 0_{m\times n}$. By the definition of matrix addition, we must have that $a_{ij} + b_{ij} = 0$ for all i and j. Solving, we get $b_{ij} = -a_{ij}$. Hence, given a matrix A, its additive inverse, which we call $-A$, does exist and is unique and, moreover, is given by the formula: $-A = [-a_{ij}]_{m\times n}$. The long and short of this is: to get the additive inverse of a matrix,

[3] A technical detail which is sadly lost on most readers.

take additive inverses of each of its entries. With the concept of additive inverse well in hand, we may now discuss what is meant by subtracting matrices. You may remember from arithmetic that $a - b = a + (-b)$; that is, subtraction is defined as 'adding the opposite (inverse).' We extend this concept to matrices. For two matrices A and B of the same size, we define $A - B = A + (-B)$. At the level of entries, this amounts to

$$A - B = A + (-B) = [a_{ij}]_{m\times n} + [-b_{ij}]_{m\times n} = [a_{ij} + (-b_{ij})]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$$

Thus to subtract two matrices of equal size, we subtract their corresponding entries. Surprised?

Our next task is to define what it means to multiply a matrix by a real number. Thinking back to arithmetic, you may recall that multiplication, at least by a natural number, can be thought of as 'rapid addition.' For example, $2 + 2 + 2 = 3 \cdot 2$. We know from algebra[4] that $3x = x + x + x$, so it seems natural that given a matrix A, we define $3A = A + A + A$. If $A = [a_{ij}]_{m\times n}$, we have

$$3A = A + A + A = [a_{ij}]_{m\times n} + [a_{ij}]_{m\times n} + [a_{ij}]_{m\times n} = [a_{ij} + a_{ij} + a_{ij}]_{m\times n} = [3a_{ij}]_{m\times n}$$

In other words, multiplying the *matrix* in this fashion by 3 is the same as multiplying *each entry* by 3. This leads us to the following definition.

Definition 8.8. Scalar[a] Multiplication: We define the product of a real number and a matrix to be the matrix obtained by multiplying each of its entries by said real number. More specifically, if k is a real number and $A = [a_{ij}]_{m\times n}$, we define

$$kA = k\,[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$$

[a]The word 'scalar' here refers to real numbers. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar).

One may well wonder why the word 'scalar' is used for 'real number.' It has everything to do with 'scaling' factors.[5] A point $P(x, y)$ in the plane can be represented by its position matrix, P:

$$(x, y) \leftrightarrow P = \begin{bmatrix} x \\ y \end{bmatrix}$$

Suppose we take the point $(-2, 1)$ and multiply its position matrix by 3. We have

$$3P = 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(-2) \\ 3(1) \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

which corresponds to the point $(-6, 3)$. We can imagine taking $(-2, 1)$ to $(-6, 3)$ in this fashion as a dilation by a factor of 3 in both the horizontal and vertical directions. Doing this to all points (x, y) in the plane, therefore, has the effect of magnifying (scaling) the plane by a factor of 3.

[4]The Distributive Property, in particular.

[5]See Section 1.7.

As did matrix addition, scalar multiplication inherits many properties from real number arithmetic. Below we summarize these properties.

Theorem 8.4. Properties of Scalar Multiplication

- **Associative Property:** For every $m \times n$ matrix A and scalars k and r, $(kr)A = k(rA)$.
- **Identity Property:** For all $m \times n$ matrices A, $1A = A$.
- **Additive Inverse Property:** For all $m \times n$ matrices A, $-A = (-1)A$.
- **Distributive Property of Scalar Multiplication over Scalar Addition:** For every $m \times n$ matrix A and scalars k and r,

$$(k + r)A = kA + rA$$

- **Distributive Property of Scalar Multiplication over Matrix Addition:** For all $m \times n$ matrices A and B scalars k,

$$k(A + B) = kA + kB$$

- **Zero Product Property:** If A is an $m \times n$ matrix and k is a scalar, then

$$kA = 0_{m\times n} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad A = 0_{m\times n}$$

As with the other results in this section, Theorem 8.4 can be proved using the definitions of scalar multiplication and matrix addition. For example, to prove that $k(A+B) = kA+kB$ for a scalar k and $m \times n$ matrices A and B, we start by adding A and B, then multiplying by k and seeing how that compares with the sum of kA and kB.

$$k(A + B) = k\left([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\right) = k\left[a_{ij} + b_{ij}\right]_{m\times n} = \left[k\left(a_{ij} + b_{ij}\right)\right]_{m\times n} = \left[ka_{ij} + kb_{ij}\right]_{m\times n}$$

As for $kA + kB$, we have

$$kA + kB = k\left[a_{ij}\right]_{m\times n} + k\left[b_{ij}\right]_{m\times n} = \left[ka_{ij}\right]_{m\times n} + \left[kb_{ij}\right]_{m\times n} = \left[ka_{ij} + kb_{ij}\right]_{m\times n} \ \checkmark$$

which establishes the property. The remaining properties are left to the reader. The properties in Theorems 8.3 and 8.4 establish an algebraic system that lets us treat matrices and scalars more or less as we would real numbers and variables, as the next example illustrates.

Example 8.3.1. Solve for the matrix A: $3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A\right) = \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix}$ using the definitions and properties of matrix arithmetic.

Solution.

$$\begin{aligned}
3A - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 9 & 12 \\ -3 & 39 \end{bmatrix} \\
3A + \left\{ - \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) \right\} &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{3}\right)(9) & \left(\frac{1}{3}\right)(12) \\ \left(\frac{1}{3}\right)(-3) & \left(\frac{1}{3}\right)(39) \end{bmatrix} \\
3A + (-1) \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + 5A \right) &= \begin{bmatrix} -4 & 2 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & 13 \end{bmatrix} \\
3A + \left\{ (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) \right\} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + (-1) \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + (-1)(5A) &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + \begin{bmatrix} (-1)(2) & (-1)(-1) \\ (-1)(3) & (-1)(5) \end{bmatrix} + ((-1)(5))A &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + (-5)A &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
3A + (-5)A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} \\
(3 + (-5))A + \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} + \left(- \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} + \left(- \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \right) \\
(-2)A + 0_{2\times 2} &= \begin{bmatrix} -1 & 6 \\ 5 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -3 & -5 \end{bmatrix} \\
(-2)A &= \begin{bmatrix} -1 - (-2) & 6 - 1 \\ 5 - (-3) & 11 - (-5) \end{bmatrix} \\
(-2)A &= \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix} \\
\left(-\frac{1}{2}\right)((-2)A) &= -\frac{1}{2} \begin{bmatrix} 1 & 5 \\ 8 & 16 \end{bmatrix} \\
\left(\left(-\frac{1}{2}\right)(-2)\right) A &= \begin{bmatrix} \left(-\frac{1}{2}\right)(1) & \left(-\frac{1}{2}\right)(5) \\ \left(-\frac{1}{2}\right)(8) & \left(-\frac{1}{2}\right)(16) \end{bmatrix} \\
1A &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -\frac{16}{2} \end{bmatrix} \\
A &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ -4 & -8 \end{bmatrix}
\end{aligned}$$

The reader is encouraged to check our answer in the original equation. □

While the solution to the previous example is written in excruciating detail, in practice many of the steps above are omitted. We have spelled out each step in this example to encourage the reader to justify each step using the definitions and properties we have established thus far for matrix arithmetic. The reader is encouraged to solve the equation in Example 8.3.1 as they would any other linear equation, for example: $3a - (2 + 5a) = -4 + \frac{1}{3}(9)$.

We now turn our attention to **matrix multiplication** - that is, multiplying a matrix by another matrix. Based on the 'no surprises' trend so far in the section, you may expect that in order to multiply two matrices, they must be of the same size and you find the product by multiplying the corresponding entries. While this kind of product is used in other areas of mathematics,[6] we define matrix multiplication to serve us in solving systems of linear equations. To that end, we begin by defining the product of a row and a column. We motivate the general definition with an example. Consider the two matrices A and B below.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

Let $R1$ denote the first row of A and $C1$ denote the first column of B. To find the 'product' of $R1$ with $C1$, denoted $R1 \cdot C1$, we first find the product of the first entry in $R1$ and the first entry in $C1$. Next, we add to that the product of the second entry in $R1$ and the second entry in $C1$. Finally, we take that sum and we add to that the product of the last entry in $R1$ and the last entry in $C1$. Using entry notation, $R1 \cdot C1 = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = (2)(3) + (0)(4) + (-1)(5) = 6 + 0 + (-5) = 1$. We can visualize this schematically as follows

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

$$\underbrace{\begin{array}{cccc} & & & \boxed{3} \\ \boxed{2} & 0 & -1 & 4 \\ & & & 5 \end{array}}_{\substack{a_{11}b_{11} \\ (2)(3)}} \begin{array}{c} + \\ + \end{array} \underbrace{\begin{array}{cccc} & & & 3 \\ 2 & \boxed{0} & -1 & \boxed{4} \\ & & & 5 \end{array}}_{\substack{a_{12}b_{21} \\ (0)(4)}} \begin{array}{c} + \\ + \end{array} \underbrace{\begin{array}{cccc} & & & 3 \\ 2 & 0 & \boxed{-1} & 4 \\ & & & \boxed{5} \end{array}}_{\substack{a_{13}b_{31} \\ (-1)(5)}}$$

To find $R2 \cdot C3$ where $R2$ denotes the second row of A and $C3$ denotes the third column of B, we proceed similarly. We start with finding the product of the first entry of $R2$ with the first entry in $C3$ then add to it the product of the second entry in $R2$ with the second entry in $C3$, and so forth. Using entry notation, we have $R2 \cdot C3 = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = (-10)(2) + (3)(-5) + (5)(-2) = -45$. Schematically,

$$\begin{bmatrix} 2 & 0 & -1 \\ -10 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{bmatrix}$$

[6] See this article on the Hadamard Product.

$$\underbrace{\begin{array}{ccc} \boxed{-10} & 3 & 5 \end{array} \quad \begin{array}{c} \boxed{2} \\ -5 \\ -2 \end{array}}_{a_{21}b_{13} = (-10)(2) = -20} \; + \; \underbrace{\begin{array}{ccc} -10 & \boxed{3} & 5 \end{array} \quad \begin{array}{c} 2 \\ \boxed{-5} \\ -2 \end{array}}_{a_{22}b_{23} = (3)(-5) = -15} \; + \; \underbrace{\begin{array}{ccc} -10 & 3 & \boxed{5} \end{array} \quad \begin{array}{c} 2 \\ -5 \\ \boxed{-2} \end{array}}_{a_{23}b_{33} = (5)(-2) = -10}$$

Generalizing this process, we have the following definition.

Definition 8.9. Product of a Row and a Column: Suppose $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times r}$. Let Ri denote the ith row of A and let Cj denote the jth column of B. The **product of R_i and C_j, denoted $R_i \cdot C_j$** is the real number defined by

$$Ri \cdot Cj = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots a_{in}b_{nj}$$

Note that in order to multiply a row by a column, the number of entries in the row must match the number of entries in the column. We are now in the position to define matrix multiplication.

Definition 8.10. Matrix Multiplication: Suppose $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times r}$. Let Ri denote the ith row of A and let Cj denote the jth column of B. The **product of A and B**, denoted AB, is the matrix defined by

$$AB = [Ri \cdot Cj]_{m \times r}$$

that is

$$AB = \left[\begin{array}{cccc} R1 \cdot C1 & R1 \cdot C2 & \ldots & R1 \cdot Cr \\ R2 \cdot C1 & R2 \cdot C2 & \ldots & R2 \cdot Cr \\ \vdots & \vdots & & \vdots \\ Rm \cdot C1 & Rm \cdot C2 & \ldots & Rm \cdot Cr \end{array} \right]$$

There are a number of subtleties in Definition 8.10 which warrant closer inspection. First and foremost, Definition 8.10 tells us that the ij-entry of a matrix product AB is the ith row of A times the jth column of B. In order for this to be defined, the number of entries in the rows of A must match the number of entries in the columns of B. This means that the number of columns of A must match[7] the number of rows of B. In other words, to multiply A times B, the second dimension of A must match the first dimension of B, which is why in Definition 8.10, $A_{m \times \underline{n}}$ is being multiplied by a matrix $B_{\underline{n} \times r}$. Furthermore, the product matrix AB has as many rows as A and as many columns of B. As a result, when multiplying a matrix $A_{\underline{m} \times n}$ by a matrix $B_{n \times \underline{r}}$, the result is the matrix $AB_{\underline{m} \times \underline{r}}$. Returning to our example matrices below, we see that A is a $2 \times \underline{3}$ matrix and B is a $\underline{3} \times 4$ matrix. This means that the product matrix AB is defined and will be a 2×4 matrix.

$$A = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right] \quad B = \left[\begin{array}{rrrr} 3 & 1 & 2 & -8 \\ 4 & 8 & -5 & 9 \\ 5 & 0 & -2 & -12 \end{array} \right]$$

[7] The reader is encouraged to think this through carefully.

Using Ri to denote the ith row of A and Cj to denote the jth column of B, we form AB according to Definition 8.10.

$$AB = \begin{bmatrix} R1 \cdot C1 & R1 \cdot C2 & R1 \cdot C3 & R1 \cdot C4 \\ R2 \cdot C1 & R2 \cdot C2 & R2 \cdot C3 & R2 \cdot C4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 & -4 \\ 7 & 14 & -45 & 47 \end{bmatrix}$$

Note that the product BA is not defined, since B is a $3 \times \underline{4}$ matrix while A is a $\underline{2} \times 3$ matrix; B has more columns than A has rows, and so it is not possible to multiply a row of B by a column of A. Even when the dimensions of A and B are compatible such that AB and BA are both defined, the product AB and BA aren't necessarily equal.[8] In other words, AB may not equal BA. Although there is no commutative property of matrix multiplication in general, several other real number properties are inherited by matrix multiplication, as illustrated in our next theorem.

Theorem 8.5. Properties of Matrix Multiplication Let A, B and C be matrices such that all of the matrix products below are defined and let k be a real number.

- **Associative Property of Matrix Multiplication:** $(AB)C = A(BC)$
- **Associative Property with Scalar Multiplication:** $k(AB) = (kA)B = A(kB)$
- **Identity Property:** For a natural number k, the $\boldsymbol{k \times k}$ **identity matrix**, denoted I_k, is defined by $I_k = [d_{ij}]_{k \times k}$ where

$$d_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

  For all $m \times n$ matrices, $I_m A = A I_n = A$.
- **Distributive Property of Matrix Multiplication over Matrix Addition:**

$$A(B \pm C) = AB \pm AC \text{ and } (A \pm B)C = AC \pm BC$$

The one property in Theorem 8.5 which begs further investigation is, without doubt, the multiplicative identity. The entries in a matrix where $i = j$ comprise what is called the **main diagonal** of the matrix. The identity matrix has 1's along its main diagonal and 0's everywhere else. A few examples of the matrix I_k mentioned in Theorem 8.5 are given below. The reader is encouraged to see how they match the definition of the identity matrix presented there.

$$\underset{I_1}{[1]} \qquad \underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{I_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

[8] And may not even have the same dimensions. For example, if A is a 2×3 matrix and B is a 3×2 matrix, then AB is defined and is a 2×2 matrix while BA is also defined... but is a 3×3 matrix!

The identity matrix is an example of what is called a **square matrix** as it has the same number of rows as columns. Note that to in order to verify that the identity matrix acts as a multiplicative identity, some care must be taken depending on the order of the multiplication. For example, take the matrix 2×3 matrix A from earlier

$$A = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right]$$

In order for the product $I_k A$ to be defined, $k = 2$; similarly, for AI_k to be defined, $k = 3$. We leave it to the reader to show $I_2 A = A$ and $AI_3 = A$. In other words,

$$\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right] = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right]$$

and

$$\left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{rrr} 2 & 0 & -1 \\ -10 & 3 & 5 \end{array} \right]$$

While the proofs of the properties in Theorem 8.5 are computational in nature, the notation becomes quite involved very quickly, so they are left to a course in Linear Algebra. The following example provides some practice with matrix multiplication and its properties. As usual, some valuable lessons are to be learned.

Example 8.3.2.

1. Find AB for $A = \left[\begin{array}{rrr} -23 & -1 & 17 \\ 46 & 2 & -34 \end{array} \right]$ and $B = \left[\begin{array}{rr} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{array} \right]$

2. Find $C^2 - 5C + 10I_2$ for $C = \left[\begin{array}{rr} 1 & -2 \\ 3 & 4 \end{array} \right]$

3. Suppose M is a 4×4 matrix. Use Theorem 8.5 to expand $(M - 2I_4)(M + 3I_4)$.

Solution.

1. We have $AB = \left[\begin{array}{rrr} -23 & -1 & 17 \\ 46 & 2 & -34 \end{array} \right] \left[\begin{array}{rr} -3 & 2 \\ 1 & 5 \\ -4 & 3 \end{array} \right] = \left[\begin{array}{rr} 0 & 0 \\ 0 & 0 \end{array} \right]$

2. Just as x^2 means x times itself, C^2 denotes the matrix C times itself. We get

$$\begin{aligned}
C^2 - 5C + 10I_2 &= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}^2 - 5\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 10 \\ -15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -10 \\ 15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 10 \\ -15 & -10 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

3. We expand $(M - 2I_4)(M + 3I_4)$ with the same pedantic zeal we showed in Example 8.3.1. The reader is encouraged to determine which property of matrix arithmetic is used as we proceed from one step to the next.

$$\begin{aligned}
(M - 2I_4)(M + 3I_4) &= (M - 2I_4)M + (M - 2I_4)(3I_4) \\
&= MM - (2I_4)M + M(3I_4) - (2I_4)(3I_4) \\
&= M^2 - 2(I_4M) + 3(MI_4) - 2(I_4(3I_4)) \\
&= M^2 - 2M + 3M - 2(3(I_4I_4)) \\
&= M^2 + M - 6I_4
\end{aligned}$$

□

Example 8.3.2 illustrates some interesting features of matrix multiplication. First note that in part 1, neither A nor B is the zero matrix, yet the product AB is the zero matrix. Hence, the the zero product property enjoyed by real numbers and scalar multiplication does not hold for matrix multiplication. Parts 2 and 3 introduce us to polynomials involving matrices. The reader is encouraged to step back and compare our expansion of the matrix product $(M - 2I_4)(M + 3I_4)$ in part 3 with the product $(x - 2)(x + 3)$ from real number algebra. The exercises explore this kind of parallel further.

As we mentioned earlier, a point $P(x, y)$ in the xy-plane can be represented as a 2×1 position matrix. We now show that matrix multiplication can be used to rotate these points, and hence graphs of equations.

Example 8.3.3. Let $R = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

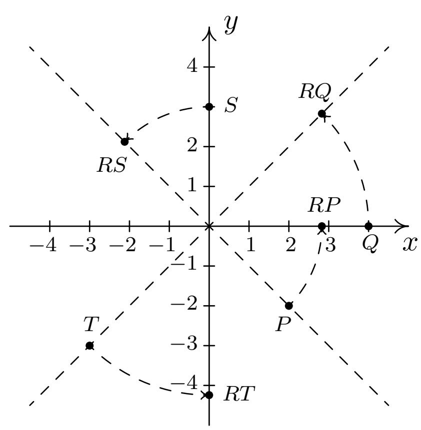
1. Plot $P(2, -2)$, $Q(4, 0)$, $S(0, 3)$, and $T(-3, -3)$ in the plane as well as the points RP, RQ, RS, and RT. Plot the lines $y = x$ and $y = -x$ as guides. What does R appear to be doing to these points?

2. If a point P is on the hyperbola $x^2 - y^2 = 4$, show that the point RP is on the curve $y = \frac{2}{x}$.

Solution. For $P(2,-2)$, the position matrix is $P = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, and

$$\begin{aligned} RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \end{aligned}$$

We have that R takes $(2,-2)$ to $(2\sqrt{2},0)$. Similarly, we find $(4,0)$ is moved to $(2\sqrt{2},2\sqrt{2})$, $(0,3)$ is moved to $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, and $(-3,-3)$ is moved to $(0,-3\sqrt{2})$. Plotting these in the coordinate plane along with the lines $y=x$ and $y=-x$, we see that the matrix R is rotating these points counterclockwise by 45°.



For a generic point $P(x,y)$ on the hyperbola $x^2 - y^2 = 4$, we have

$$\begin{aligned} RP &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{bmatrix} \end{aligned}$$

which means R takes (x,y) to $\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y, \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)$. To show that this point is on the curve $y = \frac{2}{x}$, we replace x with $\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and y with $\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ and simplify.